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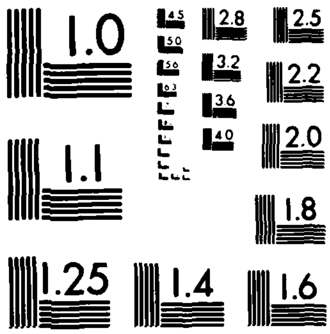

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Los Angeles

Flutter Control
with Unsteady Aerodynamic Models

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Engineering

by

Shyang Chang

1984

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This dissertation deals with a generic problem for aircraft: control laws for flutter suppression. Until recently, the system frequency response was approximated by rational functions so that the finite-dimensional L-Q-R theory could be applied. However, discrepancies between theory and practice, especially in transient response, has led to renewed interest in the problem.

It would appear the the L-Q-R theory would need infinite dimensional state space models. In this research, we first develop a time-domain model for unsteady aerodynamic loads and then couple it with a lumped model for the structural dynamics. We show that the solution to the resulting input-output system, characterized by integro-differential equations, can be endowed with a state space which is a reflexive Banach space, and the state equations have a unique semigroup solution. We go on to examine the input-output stability for such a system. We show that input-output stability need not imply stability of the states. By a suitable approximation of the Sears function near the origin, we show that the (CONTINUED)

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ITEM #19, ABSTRACT, CONTINUED: infinite dimensional (L_2) L-Q-R theory can be applied. We derive optimal feedback control laws ensuring "weak" stability of the states, as well as input-output stability.

The dissertation of Shyang Chang is approved.

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1984

TO TAI-YUN, YUO-FONG, YUO-LING

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ABSTRACT OF THE DISSERTATION

Flutter Control
with Unsteady Aerodynamic Models

by

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Doctor Of Philosophy in Engineering
University of California, Los Angeles, 1984
Professor A.V. Balakrishnan, Chairman

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endowed with a state space which is a reflexive Banach space, and the state equations have a unique semigroup solution. We go on to examine the input-output stability for such a system. We show that input-output stability need not imply stability of the states. By a suitable approximation of the Sears function near the origin, we show that the infinite dimensional (L_2) L-Q-R theory can be applied. We derive optimal feedback control laws ensuring "weak" stability of the states, as well as input-output stability.

CHAPTER 1

INTRODUCTION

This dissertation is in the general area of modelling and control of aircraft flutter problem. Unsteady aerodynamics is also studied because of independent interest.

NATURE OF THE FLUTTER PROBLEM

Aerodynamic flutter refers to a subject that has developed from the earliest days of manned flight. Flutter is an unstable motion due to the interaction between structural vibrations and the aerodynamic forces which results in the extraction of energy from the air. It occupies a prime role in current design in the whole spectrum of advanced aircraft, missiles and spacecraft. The field is also one of great inherent interest as a scientific and technical discipline.

The techniques required involve the study of unsteady aerodynamics for arbitrary motions, structural dynamics due to unsteady loading, and aerodynamic loading caused by control surface motion. The primary design goal is structural stability. Hence, this dissertation will focus on the modelling of unsteady aerodynamics, coupled aerostructural motion and flutter control systems.

PREVIOUS WORKS

Traditional methods of flutter analysis have proceeded in three steps: first determining the vibration modes of the structure without aerodynamic forces present; then calculating the aerodynamic forces on the wing due to simple harmonic oscillations of the normal modes as functions of Mach number, altitude, and reduced frequency; and finally calculating the flutter boundary.

J.W. Edwards [1] and Edwards et al. [2] have extended the unsteady aerodynamic theory from simple harmonic oscillations to arbitrary motions using Laplace transform techniques. H. Ashley et al. [3] and E.H. Dowell et al. [4] present similar formulations involving inverse Fourier transforms to obtain impulse response function airloads. These formulations are mathematically correct, but the calculations are cumbersome and involve functions available only in tabulated form. Hence few examples of the exact transient response of airfoils excited by control surface motion have been calculated.

The first system theoretic formulation of the problem is due to Balakrishnan [18]. Following his work, Burns et al [6] have also introduced an infinite-dimentional state space, although their approach is based on retarded functional differential equations.

OBJECTIVES AND CONTRIBUTIONS OF THIS STUDY

In order to provide a basis for the analysis of aeroelastic control schemes, we develop the exact transient response of a three degree-of-freedom airfoil. The development makes extensive use of special time-domain functions derived from a function studied by Kussner (Sears, [5]).

Currently the active control of flutter in flexible flight vehicles has gone beyond the frequency domain analysis in an effort to apply linear quadratic regulator theory to the problem. However most of the works are confined to finite dimensional theory via rational or Pade approximations.

The major problems with those approaches are that (1) the control obtained via finite dimensional approximation is not put back into the original system; (2) the singularity near the origin will not be seen after rational approximation.

It is apparent that the problem could not be solved without the conscientious introduction of infinite dimensional state space. So the first step is to develop a state space model for motion of an airfoil in two-dimensional unsteady flow, of an inviscid, incompressible fluid. Then we show that the solution exists, is unique and depends continuously on the initial data. Of particular importance in the context of the present work is the fact that we examine input-output statility and L-Q-R theory for such system and thus may provide appro-prite design tools for flutter suppression problems.

OUTLINE OF DISSERTATION

The mathematical model of unsteady aerodynamics is presented in Chapter 2. The interaction of aerodynamics with structure motion is given in Chapter 3. The complete equations of motion are obtained as a set of coupled, integro-differential equation. In Chapter 4 a state space model is derived from the integro-differential equation and the well-posedness of the model is proved. Chapter 5 is devoted to input-output stability and L-Q-R theory. The final chapter presents the conclusion of this research and suggestions for future research.

CHAPTER 2

MATHEMATICAL MODEL OF UNSTEADY AERODYNAMICS

UNSTEADY AERODYNAMICS

The fundamental equations of unsteady aerodynamics follow from the equation of state and conservations of mass, momentum and energy. The derivation is presented in numerous books. We follow the notation, nomenclature and development of Bisplinghoff, et al. [3].

The exact equation satified by the velocity potential is

$$\nabla^2 \phi - \frac{1}{a^2}\left[\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial q^2}{\partial t} + \underline{q} \cdot \nabla \left(\frac{q^2}{2} \right)\right] = 0 \quad (2.1)$$

where the velocity vector is given by

$$\underline{q} = \nabla \phi \quad . \quad (2.2)$$

And the relationship between ϕ and a is given by the unsteady Bernoulli's equation

$$\frac{a^2 - a_\infty^2}{\gamma - 1} = \frac{U_\infty^2}{2} - \left(\frac{\partial \phi}{\partial t} + \frac{q^2}{2} \right) \quad . \quad (2.3)$$

The boundary conditions associated with this pair of partial differential equations are

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \underline{q} \cdot \nabla F = 0 \quad (2.4)$$

and Kutta condition for trailing edges of wings.

The general nonlinear equations of potential flow are difficult to solve. In order to be able to obtain solutions to these equations, it is necessary to use small perturbation theory.

Linearization is obtained by assuming that the body is thin, so that the velocity vector varies only slightly from the free-stream velocity. A disturbance velocity potential $\hat{\phi}$ is defined such that

$$\phi = Ux + \hat{\phi} \quad .$$

Then the linearlized partial differential equations for unsteady, compressible flow are (we henceforward drop the ^ on ϕ, a)

$$\nabla^2\phi - \frac{1}{a_\infty^2}\left[\frac{\partial^2\phi}{\partial t^2} + 2U \frac{\partial^2\phi}{\partial x \partial t} + U^2 \frac{\partial^2\phi}{\partial x^2} \right] = 0 \qquad (2.5)$$

$$a = -\frac{\gamma - 1}{2a_\infty}\left[\frac{\partial\phi}{\partial t} + U \frac{\partial\phi}{\partial x} \right] \qquad (2.6)$$

subject to <u>linearized</u> boundary conditions.

TWO-DIMENSIONAL, INCOMPRESSIBLE UNSTEADY AERODYNAMICS

As a first step, we confine ourselves to incompressible, inviscid flow only. We will derive a time-domain model from the basic aerodynamic equations and appropriate boundary conditions (see Balakrishnan [7]).

Consider a typical section as shown in Fig. 2.1, extending along the X-axis from -1 to +1, with motion entirely in the X-Z plane. Then the disturbance velocity potential $\phi(t,x,z)$ is given by the Laplace equation:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad , \quad t > 0 \; ; \; |x| > 1 \qquad (2.7)$$

with the boundary conditions:

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=0^{+,-}} = W_a(t,x) \qquad |x| < 1, \quad t > 0 \qquad (2.8)$$

$$\frac{\partial\phi(t,x,0)}{\partial t} + U\,\frac{\partial\phi(t,x,0)}{\partial x} = 0 \quad 1 \le x \le 1+Ut \;, t > 0 \; (2.9)$$

$$\frac{\partial\phi(t,1,0)}{\partial t} + U\,\frac{\partial\phi(t,1,0)}{\partial x} = 0 \qquad t > 0 \quad . \qquad (2.10)$$

The boundary conditions (2.8), (2.9) and (2.10) are the flow-tangency condition, Zero-pressure-discontinuity and Kutta conditions, respectively.

Following Schwarz (see Bisplinghoff, et al. [3]), it is easy to check that the $\phi(t,x,z)$ below satisfies equation (2.7).

$$\phi(t,x,z) = -\frac{1}{2\pi}\left[\int_{-1}^{1}\gamma_a(t,s)\,\tan^{-1}\frac{z}{x-s}\,ds + \right.$$

$$+ \int_{1}^{1+Ut} \gamma_w(t,s) \tan^{-1} \frac{z}{x-s}\, ds \Big], \quad (2.11)$$

where γ_a (t,x) is the circulation on the foil, and γ_w (t,x) the circulation in the wake. They are to be determined from the given boundary conditions. The integrals are to be taken in the Cauchy sense.

The nest step is to represent ϕ in terms of the downwash function W_a (t,x) through the given boundary conditions.

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=0^{+},-} = W_a(t,x) = -\frac{1}{2\pi}\left[\int_{-1}^{1} \frac{\gamma_a(t,\xi)}{x-\xi}\, d\xi + \int_{1}^{1+Ut} \frac{\gamma_w(t,\xi)}{x-\xi}\, d\xi\right]. \quad (2.12)$$

Moreover,

$$\frac{\partial \phi}{\partial x}(t,x,0^{+}) = \frac{1}{2}\gamma_a\ (t,x) \qquad -1 < x < 1$$

$$= \frac{1}{2}\gamma_w\ (t,x) \qquad 1 < x < 1+Ut \qquad (2.13)$$

$$\frac{\partial \phi}{\partial x}(t,x,0^{+}) = 0 \qquad x < -1,\ x > 1+Ut \quad (2.14)$$

and that

$$- \frac{\partial \phi}{\partial x}(t,x,0^-) = \frac{\partial \phi}{\partial x}(t,x,0^+) \quad . \tag{2.15}$$

Hence

$$\phi(t,x,0^+) = \frac{1}{2}\int_{-1}^{x} \gamma_a \, (t,\xi)d\xi \quad , -1 < x < 1$$

$$= \frac{1}{2} \left\{ \int_{-1}^{1} \gamma_a \, (t,\xi)d\xi + \int_{1}^{x} \gamma_w \, (t,\xi)d\xi \right\},$$

$$1 < x < 1+Ut \tag{2.16}$$

Define

$$\Gamma(t) \equiv \int_{-1}^{1} \gamma_a \, (t,\zeta)d\zeta \quad . \tag{2.17}$$

It is clear from (2.11) that

$$\phi(t,x,-z) = -\phi(t,x,z,)$$

$$\frac{\partial \phi}{\partial t}(t,x,-|z|) = -\frac{\partial \phi}{\partial t}(t,x,|z|).$$

In particular,

$$\frac{\partial \phi}{\partial t}(t,x,0^+) = -\frac{\partial \phi}{\partial t}(t,x,0^-) \quad .$$

Hence the zero-pressure-discontinuity boundary condition yields:

$$\frac{1}{2}\left[\Gamma'(t)+\int_1^x \frac{\partial}{\partial t}\gamma_w(t,\zeta)d\zeta + U\gamma_w(t,x)\right] = 0 \quad ,$$

$$1 \le x \le 1+Ut$$

$$U\gamma_w(t,1^+) + \Gamma'(t) = 0 \quad , \tag{2.18}$$

and Kutta condition yields

$$U\gamma_a(t,-1) + \Gamma'(t) = 0 \quad .$$

Therefore,

$$\frac{-\Gamma'(t)}{U} = \gamma_a(t,-1) = \gamma_w(t,1^+) \text{ and is finite.}$$

From (2.18), $\gamma_w(t,x)$ must satisfy:

$$\gamma_w(t,x) = F\left(t - \frac{x}{U}\right) \quad . \tag{2.19}$$

Putting $x = 1^+$,

$$\gamma_w(t,1^+) = F\left(t - \frac{1}{U}\right) = -\frac{\Gamma'(t)}{U}$$

$$\text{and } F(t) = -\frac{\Gamma'(t+\frac{1}{U})}{U} \quad , \; t \ge 0 . \tag{2.20}$$

Therefore,

$$\gamma_w(t,x) = -\frac{1}{U}\Gamma'\left(t + \frac{1-x}{U}\right) \quad . \tag{2.21}$$

Substitute (2.21) into (2.12), we get

$$W_a\ (t,x) = -\frac{1}{2\pi}\left[\int_{-1}^{1} \frac{\gamma_a(t,\sigma)}{x-\sigma}\, d\sigma \right.$$

$$\left. - \frac{1}{U}\int_{1}^{1+Ut} \frac{\Gamma'(t+\frac{1-\sigma}{U})}{x-\sigma}\, d\sigma\right] ,$$

or $$\frac{1}{2\pi}\int_{-1}^{1} \frac{\gamma_a(t,\sigma)}{x-\sigma}\, d\sigma = -W_a\ (t,x)$$

$$+ \frac{1}{2}\int_{1}^{1+Ut} \frac{\Gamma'(t+\frac{1-\sigma}{U})}{(x-\sigma)U}\, d\sigma$$

$$= -W_a\ (t,x)$$

$$+ \frac{1}{2}\int_{0}^{t} \frac{\Gamma'(t-y)}{x-1-Uy}\, dy \quad ,$$

$$-1 < x < 1 \quad . \quad (2.22)$$

Since $\gamma_a\ (t,1^-)$ is finite, Söhngen [8] proves that this integral equation has a unique solution.

$$\gamma_a\,(t,x) = -\frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\left\{\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\left(-W_a\,(t,\zeta)\right.\right.$$

$$\left.\left.+\ \frac{1}{2\pi}\int_0^t \frac{\Gamma'(t-y)}{\zeta-1-Uy}\,dy\right)\frac{1}{x-\zeta}\,d\zeta\right\}$$

$$= \frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\left\{\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\,\frac{W_a(t,\zeta)}{x-\zeta}\,d\zeta\right.$$

$$-\ \frac{1}{2\pi}\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\,\frac{d\zeta}{x-\zeta}$$

$$\left.\cdot\ \int_0^t \frac{\Gamma'(t-y)}{\zeta-1-Uy}\,dy\right\} \qquad (2.23)$$

This is our basic integral equation.

Let $$H\,(\sigma,x) = \int_{-1}^{1}\sqrt{\frac{1+y}{1-y}}\,\frac{1}{y-1-U\sigma}\cdot\frac{1}{x-y}\,dy \qquad (2.24)$$

and $$H\,(\sigma) = \frac{2}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\;H\,(\sigma,x)\,dx\ . \qquad (2.25)$$

Then the equation becomes (Balakrishnan and Edward, [6])

$$\gamma_a (t,x) = \frac{2}{\pi}\sqrt{\frac{1-x}{1+x}} \left\{ \int_{-1}^{1}\sqrt{\frac{1+y}{1-y}} \frac{W_a(t,y)}{x - y} \right.$$

$$\left. - \frac{1}{2\pi} \int_0^t H(\eta,x)\, \Gamma'(t-\eta)d\eta \right\} , \qquad (2.26)$$

where

$$\Gamma(t) = -2 \int_{-1}^{1}\sqrt{\frac{1+y}{1-y}}\, W_a (t,y)dy - \frac{1}{2\pi} \int_0^t H(\eta)\Gamma'(t-\eta)d\eta .$$

$$\ldots(2.27)$$

Substituting (2.26), (2.21) into (2.11) we obtain the solution for the disturbance velocity potential.

CIRCULATION ON THE AIRFOIL

With reference to fig. 2.1, let $h(t)$ denote the plunge coordinate, $\alpha(t)$ the angle of attack, and $\beta(t)$ the flap deflection. Then the downwash function $W_a (t,x)$ is given by

$$W_a (t,x) = \frac{\partial}{\partial t} z_a (t,x) + U \frac{\partial}{\partial x} z_a (t,x) , \qquad (2.28)$$

where

$$z_a(t,x) = -h(t) - (x-a)\alpha(t), \qquad -1 < x < c$$

$$= -h(t) - (x-a)\alpha(t) - (x-c)\beta(t),$$

$$c < x < 1 \, . \qquad (2.29)$$

Then

$$W_a(t,x) = -h'(t) - (x-a)\alpha'(t) - U\alpha(t),$$

$$-1 < x < c$$

$$= -h'(t) - (x-a)\alpha'(t) - (x-c)\beta'(t) - U\alpha(t)$$

$$-U\beta(t) \, , \qquad c < x < 1 \, . \qquad (2.30)$$

Hence

$$\Gamma(t) = \int_{-1}^{1} \gamma_a(t,x)dx$$

$$= \frac{2}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,\frac{dx}{x-\zeta}\left\{\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\,W_a(t,\zeta)d\zeta\right.$$

$$\left. - \frac{1}{2\pi}\int_0^t H(\tau)\Gamma'(t-\tau)d\tau\right\}$$

$$= -2\int_{-1}^{1}\sqrt{\frac{1+\sigma}{1-\sigma}}\,W_a(t,\sigma)\,d\sigma -$$

$$- \frac{1}{2\pi} \int_0^t H(\sigma)\Gamma'(t-\sigma)d\sigma. \tag{2.31}$$

Substitute (2.30) into (2.31), we get

$$\Gamma(t) = 2\pi U\alpha(t) + \left(2\cos^{-1}c + 2\sqrt{1-c^2} \right)U\beta(t) + 2\pi h'(t)$$

$$+ \left(2\sqrt{1-c^2} + (1-2c)\cos^{-1}c - c\sqrt{1-c^2} \right)\beta'(t)$$

$$+ (\pi - 2\pi a)\alpha'(t) + \int_0^t \left(1-\sqrt{\frac{z+1}{z-1}} \right)\Gamma'(t-\sigma)d\sigma.$$

...(2.32)

Let

$$x(t) = \begin{bmatrix} h(t) \\ \alpha(t) \\ \beta(t) \end{bmatrix} \text{ and } \quad Z(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix},$$

then

$$\Gamma(t) = [\, B \,,\, Z(t) \,] + \int_0^t \left(1-\sqrt{\frac{z+1}{z-1}} \right)\Gamma'(t-\sigma)d\sigma,$$

...(2.33)

where

$$B = \begin{bmatrix} 0 \\ 2\pi U \\ (\ 2\cos^{-1}c + 2\sqrt{1-c^2}\ )\ U \\ 2\pi \\ \pi - 2\pi a \\ (\ 1 - 2c\ )\cos^{-1}c + (\ 2 - c\ )\sqrt{1-c^2} \end{bmatrix} .$$

But

$$\Gamma(0) = [B,\ Z(0)]$$

Hence

$$0 = [B,\ Z(t)-Z(0)] - \int_0^t \sqrt{\frac{z+1}{z-1}}\,\Gamma'\ (t-\sigma)d\sigma . \qquad (2.34)$$

For

$\operatorname{Re} s > 0$:

$$\int_0^\infty e^{-s\sigma}\sqrt{\frac{z+1}{z-1}}\,d\sigma = \frac{1}{U}\int_1^\infty e^{-\frac{s}{U}(z-1)}\sqrt{\frac{z+1}{z-1}}\,dz$$

$$= \frac{1}{U}\,e^{\frac{s}{U}}\int_1^\infty e^{-\frac{sz}{U}}\sqrt{\frac{z+1}{z-1}}\,dz . \qquad (2.35)$$

Let

$$K_0(s) = \int_1^\infty e^{-st}\frac{dt}{\sqrt{t^2-1}} .$$

Then

$$-K_0'(s) = \int_1^\infty e^{-st} \frac{t\,dt}{\sqrt{t^2-1}} .$$

Hence

$$\int_1^\infty e^{\frac{-sz}{U}} \sqrt{\frac{z+1}{z-1}}\, dz = K_0\left(\frac{s}{U}\right) - K_0'\left(\frac{s}{U}\right), \tag{2.36}$$

and

$$\int_0^\infty e^{-s\sigma} \sqrt{\frac{z+1}{z-1}}\, d\sigma = \frac{1}{U} e^{\frac{s}{U}} \left(K_0\left(\frac{s}{U}\right) - K_0'\left(\frac{s}{U}\right)\right),$$

$$\int_0^\infty e^{-s\sigma}\, \Gamma'(\sigma) d\sigma = \frac{U\left[B, L(Z;s) - \frac{Z(0)}{s}\right] e^{-\frac{s}{U}}}{\left(K_0\left(\frac{s}{U}\right) - K_0'\left(\frac{s}{U}\right)\right)} ,$$

where

$$L(Z;s) = \int_0^\infty e^{-st}\, Z(t)\, dt . \tag{2.37}$$

We will define $c_1(t)$ as the inverse Laplace transform of

$$\frac{1}{s} \frac{U e^{-\frac{s}{U}}}{K_0\left(\frac{s}{U}\right) - K_0'\left(\frac{s}{U}\right)} .$$

Then

$$\Gamma'(t) = \int_0^t c_1(t-\sigma)\, [B, \dot{Z}(\sigma)]\, d\sigma \tag{2.38}$$

and

$$\Gamma(t) = \int_0^t c_1(t-\sigma)[B,Z(\sigma)]d\sigma - c_2(t)[B,Z(0)]$$

$$+ [B,Z(0)] , \qquad (2.39)$$

where

$$c_2(t) = \int_0^t c_1(s)\, ds .$$

Finally we note the series expanision due to Küssner (Sears, [5]) for $c_1(t)$:

$Ut \leq 5$

$$c_1(t) = U\frac{\sqrt{2}}{\pi}\left\{\frac{1}{2}(Ut)^{-\frac{1}{2}} - \frac{1}{8}(Ut)^{\frac{1}{2}} + \frac{5}{192}(Ut)^{\frac{3}{2}} - \frac{161}{26880}(Ut)^{\frac{5}{2}} \ldots\right\} . \qquad (2.40)$$

$Ut \geq 5$

$$c_1(t) = \frac{U}{(Ut-1)^2}\left\{1 - \frac{5}{(Ut-1)} + \frac{4\mathrm{Log}(2Ut-2)}{Ut-1} - \frac{54\mathrm{Log}(2Ut-2)}{(Ut-1)^2} + \frac{(14+\frac{9}{2}-3\pi^2)}{(Ut-1)^2} + \frac{18(\mathrm{Log}(2Ut-2))^2}{(Ut-1)^2} \ldots\right\} . \qquad (2.41)$$

CALCULATION OF THE LIFT, PITCHING MOMENT AND FLAP MOMENT

In this section, we will write down the lift, pitching moment and flap moment expressions. The detailed derivations are in Appendices A, B, C.

1. $$P = (-\rho)\Big\{ U\Gamma(t) + U\pi\alpha'(t) - U\,(c\sqrt{1-c^2} - \cos^{-1}c)$$

$$\cdot\ \beta'(t) + \pi h''(t) - a\pi\alpha''(t) - (\tfrac{1}{3}(\sqrt{1-c^2})^3$$

$$+ c\cos^{-1}c - \sqrt{1-c^2})\,\beta''(t) -$$

$$\frac{d}{dt}\int_0^t c_3\,(t-\epsilon)\,\big[B,\dot{Z}(\epsilon)\big]\,d\epsilon \Big\}, \qquad (2.42)$$

where

$$c_3(t) = \int_0^t c_1(t-\epsilon)\,\Big[U\epsilon - \sqrt{U^2\epsilon^2 + 2U\epsilon}\Big]d\epsilon\ . \qquad (2.43)$$

2. $$M_\alpha = (-\rho)\{ U^2(-1-2a)\pi\alpha(t) + U^2\,\big[(-1-2a)\cos^{-1}c$$

$$+ (c-2a)\sqrt{1-c^2}\,\big]\beta(t) - U\,(1+2a)\pi h'(t)$$

$$+ U\pi\,(a^2-a-\tfrac{1}{2})\,\alpha'(t) + U\,\big[(-\tfrac{5}{6}+3ac-2a-2a^2-\tfrac{c^3}{3}-\tfrac{2c^3}{3})$$

$$\cdot\ \sqrt{1-c^2} + (2ac+c-\tfrac{1}{2}-2a-a^2)\cos^{-1}c\big]\,\beta'(t) -$$

$$- (\tfrac{1}{2}+a^2+a)\pi h''(t) - (\tfrac{1}{8}-\tfrac{a}{2}-a^3-\tfrac{a^2}{2})\pi\alpha''(t) -$$

$$- \Big[\Big(\frac{1}{8}+\frac{a^2}{2}-\frac{c}{2}-a^2c-ac\Big)\cos^{-1}c+\sqrt{1-c^2}\Big(\frac{1}{2}+a^2+\frac{2}{3}a+\frac{1}{3}ac^2$$

$$-\frac{a^2c}{2}-\frac{c^3}{12}-\frac{c}{24}\Big)\Big]\ \beta''(t)+\int_0^t\Big[\frac{3}{4}\ c_1(t-\sigma)\ +\ Uc_4(t-\sigma)$$

$$+Ua(1-c_2(t-\sigma))\Big]\ [B,Z(\sigma)]\ d\sigma\ +\frac{d}{dt}\int_0^t\Big[\Big(\frac{1}{4}+\frac{a^2}{2}\Big)$$

$$-\frac{1}{4}\ c_2(t-\sigma)\ +\ ac_3(t-\sigma)+\ \frac{1}{2}\ c_5(t-\sigma)\Big]\ [B,Z(\sigma)]\quad d\sigma\Big\},$$

$$\ldots(2.44)$$

where

$$c_4(t) = c_2(t) + c_3(t)$$

$$c_5(t) = \int_0^t c_1(t-\sigma)\cdot\Big[(1+U\sigma)\sqrt{U^2\sigma^2+2U\sigma}\ -\ (1+U\sigma)^2\Big]d\sigma\ .$$

$$\ldots(2.45)$$

3. $$M_\beta = (-\rho)\ \Big\{\ U^2\big[(2+c)\sqrt{1-c^2}\ -\ (2c+1)\cos^{-1}c\ \big]\ \alpha(t)$$

$$-\ U^2f_1(c)\beta(t)\ +\ U\ \big[\ (2+c)\sqrt{1-c^2}\ -\ (2c+1)\cos^{-1}c\big]$$

$$\cdot\ h'(t)\ +\ U\ \Big[\Big(\ \frac{8}{3}+c+\frac{c^2}{3}\Big)\sqrt{1-c^2}\ -\ (1+3c)\cos^{-1}c$$

$$-\ h_1(c)\Big]\ \alpha'(t)\ +\ U\Big[\ \frac{g_1(c)}{2}\ -\ (f_2(c)-cf_1(c))\Big]\ \beta'(t)-$$

$$- h_1(c)h''(t) + (ah_1(c)-h_2(c))\,\alpha''(t) + \frac{1}{2}\,(g_2(c) -$$

$$- cg_1(c))\,\beta''(t) - \frac{U}{2\pi}\int_0^t H(\eta)\Gamma'(t-\eta)d\eta + \frac{1}{4\pi}$$

$$\cdot \frac{d}{dt}\int_0^t H_2(\eta)\,\Gamma'(t-\eta)d\eta \quad , \qquad (2.46)$$

where

$$f_1(c) = \frac{1}{\pi}\left[(2c+1)(\cos^{-1}c)^2 - (1-c^2) - 2\sqrt{1-c^2}\cos^{-1}c\right]$$

$$f_2(c) = -\frac{1}{\pi}\left\{\left[(1+c)\cos^{-1}c + (c^2-c-2)\sqrt{1-c^2})^3\right]\right.$$

$$\left.\cdot(\cos^{-1}c + \sqrt{1-c^2}) + \frac{1}{3}(\sqrt{1-c^2})^3\cos^{-1}c\right\}$$

$$g_1(c) = \sqrt{1-c^2}\left[\frac{c\sqrt{1-c^2}}{2\pi} - \frac{(1+2c^2)}{3\pi}\cos^{-1}c + \frac{1}{3\pi}(2c^2\right.$$

$$\left.+9c+4)\right] - \frac{1}{\pi}(\cos^{-1}c)^2(1-2c+2c^2)$$

$$g_2(c) = (\cos^{-1}c + \sqrt{1-c^2})(-\frac{1}{\pi})\left[(1-2c+2c^2)\cos^{-1}c -\right.$$

$$\left.\frac{1}{3}(2c^2+9c+4)\sqrt{1-c^2}\right] - \frac{2}{\pi}\left[(c+\frac{c^2}{2}-\frac{3}{8})\cos^{-1}c + (\frac{7c^3}{4} -\right.$$

$$\left.\frac{5}{16}c)\sqrt{1-c^2}\right] - \frac{1}{6\pi}\left[c^2\sqrt{1-c^2} - \frac{5}{2}c\sqrt{1-c^2}\cos^{-1}c + \right.$$

$$+\ (2+\frac{c^4}{2}-\frac{5c^2}{2})]$$

$$h_1(c) = \left(\frac{1-2c-2c^2}{2}\right)\cos^{-1}c - \frac{1}{6}(2c^2-9c-4)\sqrt{1-c^2}$$

$$h_2(c) = \left(\frac{c^2}{2}+\frac{1}{8}\right)\cos^{-1}c - \left(\frac{c^3}{12}-\frac{13}{24}c\right)\sqrt{1-c^2}\ .$$

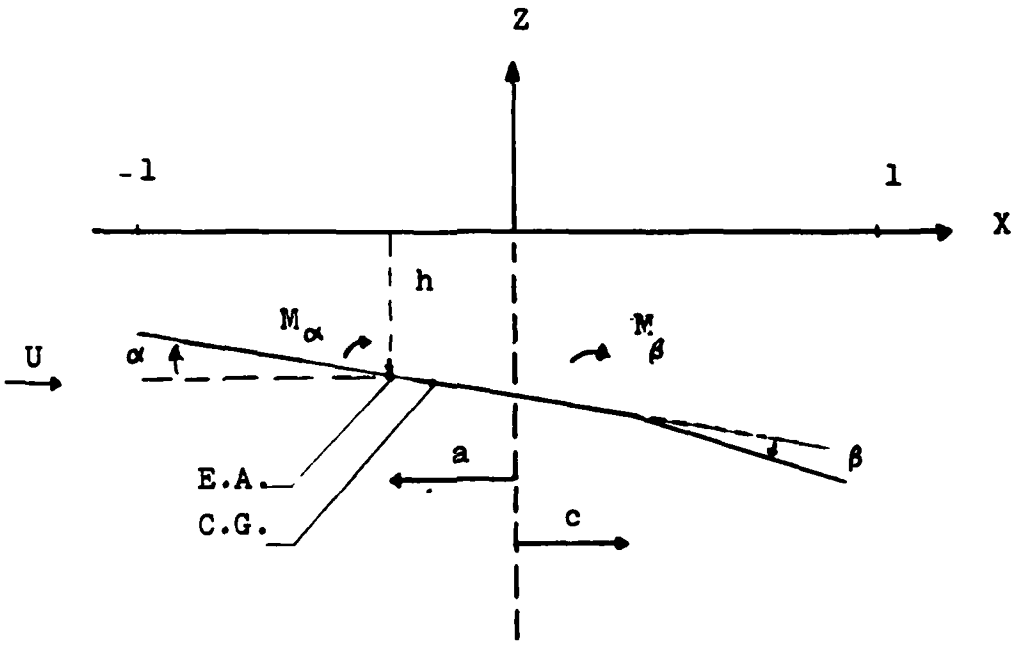

Fig.2.1

CHAPTER 3

COUPLED AERO-STRUCTURAL DYNAMICS

In aeroelasticity the aircraft structures under consideration are perfectly elastic. That is, the model describing the structure requires partial differential equations. As a first step, we will only treat a three-degree-freedom two-dimensional typical sections which may be regarded as representing the first bending and torsion modes of a three-dimensional flexible wing.

EQUATIONS OF MOTION OF A TYPICAL SECTION

The equations of motion (following the usual conventions, see Theodorsen [9], and Edwards [1]) can be written:

$$M_s\ddot{x} + B_s\dot{x} + K_s x = \frac{L}{m_s} + Gu \quad , \tag{3.1}$$

where the subscript s stands for structure, and

$$M_s = \begin{bmatrix} 1 & x_\alpha & x_\beta \\ x_\alpha & \gamma_\alpha^2 & [\gamma_\beta^2 + x_\beta(c-a)] \\ x_\beta & \gamma_\beta^2 + x_\beta(c-a) & \gamma_\beta^2 \end{bmatrix}$$

$$K_s = \begin{bmatrix} \omega^2 & 0 & 0 \\ 0 & \gamma_\alpha^2\omega_\alpha^2 & 0 \\ 0 & 0 & \gamma_\beta^2\omega_\beta^2 \end{bmatrix}$$

$$B_s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\gamma_\beta^2 \zeta_\beta \omega_\beta \end{bmatrix}$$

$$G = \begin{bmatrix} 0 \\ 0 \\ \gamma_\beta^2 \omega_\beta^2 \end{bmatrix}$$

$$L = \begin{bmatrix} P \\ M_\alpha \\ M_\beta \end{bmatrix}$$

$$P = P_c + P_{nc}$$

$$M_\alpha = M_{\alpha,c} + M_{\alpha,nc}$$

$$M_\beta = M_{\beta,c} + M_{\beta,nc}$$

where the subscript c stands for " circulatory" and nc for "non-circulatory". For non-circulatory part,

$$\begin{bmatrix} P_{nc} \\ M_{\alpha,nc} \\ M_{\beta,nc} \end{bmatrix} = M_a \ddot{x} + B_a \dot{x} + K_a x \ ,$$

where the subscript "a" stands for aerodynamics .

$$K_a = -\frac{\rho}{m_s}\begin{bmatrix} 0 & 0 & 0 \\ 0 & U^2(-1-2a)\pi & \pi U^2[(-1-2a)\cos^{-1}c + (c-2a)\sqrt{1-c^2}] \\ 0 & U^2[(2+c)\sqrt{1-c^2} - (1+2c)\cos^{-1}c] & -U^2 f_1(c) \end{bmatrix}$$

$$B_a = -\frac{\rho}{m_s}\begin{bmatrix} 0 & \pi U & -U[c\sqrt{1-c^2} - \cos^{-1}c] \\ -\pi U(1+2a) & \pi U(a^2-a-\frac{1}{2}) & U\{(-\frac{5}{6}+3ac-2a-2a^2-\frac{c^2}{3}-\frac{2c^3}{3})\sqrt{1-c^2} - (\frac{1}{2}-a^2+2a-2ac-c)\cos^{-1}c\} \\ U[(2+c)\sqrt{1-c^2} - (1+2c)\cos^{-1}c] & U[(\frac{8}{3}+c+\frac{c^2}{3})\cdot\sqrt{1-c^2} - (1+3c)\cos^{-1}c - h_1(c)] & U[\frac{g_1(c)}{2} - (f_2(c) - c\cdot f_1(c))] \end{bmatrix}$$

$$M_a = -\frac{\rho}{m_s}\begin{bmatrix} \pi & -a\pi & \sqrt{1-c^2} - c\cos^{-1}c - \frac{1}{3}\left(\sqrt{1-c^2}\right)^3 \\ \pi\left(-\frac{1}{2}-a^2-a\right) & \pi\left(a^3+\frac{a^2}{2}+\frac{a}{2}-\frac{1}{8}\right) & -\left(\frac{1}{8}+\frac{a^2}{2}-\frac{c}{2}-a^2c-ac\right)\cos^{-1}c - 2\sqrt{1-c^2}\left(\frac{1}{4}+\frac{a^2}{2}+\frac{a}{3}+\frac{ac^2}{6}-\frac{a^2c}{4}-\frac{c^3}{24}-\frac{c}{48}\right) \\ -h_1(c) & ah_1(c)-h_2(c) & \frac{1}{2}\left(g_2(c) - cg_1(c)\right) \end{bmatrix}$$

Finally:

$$\begin{bmatrix} P_c \\ M_{\alpha,c} \\ M_{\beta,c} \end{bmatrix} = -\frac{\rho}{m_s}\begin{bmatrix} U\int_0^t c_2(t-\xi)\left[B,\dot{Z}(\xi)\right]d\xi + U\left[B,Z(0)\right] - \frac{d}{dt}\int_0^t c_3(t-\xi)\left[B,\dot{Z}(\xi)\right]d\xi \\ \int_0^t\left(\frac{3}{4}c_1(t-\xi) + Uc_4(t-\xi) + Ua(1-c_2(t-\xi))\right)\left[B,\dot{Z}(\xi)\right]d\xi + \frac{d}{dt}\int_0^t\left(\left(\frac{a^2}{2}+\frac{1}{4}\right) - \frac{1}{4}c_2(t-\xi) + ac_3(t-\xi) + \frac{1}{2}c_5(t-\xi)\right)\left[B,\dot{Z}(\xi)\right]d\xi \\ -\frac{U}{2\pi}\int_0^t c_6(t-\xi)\left[B,\dot{Z}(\xi)\right]d\xi + \frac{1}{2}\frac{d}{dt}\frac{1}{2\pi}\int_0^t c_7(t-\xi)\left[B,\dot{Z}(\xi)\right]d\xi \end{bmatrix} \quad \ldots(3.2)$$

where

$$c_6(t)=\int_0^t c_1(t-\xi)H_1(\xi)d\xi$$

$$c_7(t)=\int_0^t c_1(t-\xi)H_2(\xi)d\xi .$$

Moreover, we can write (3.2) in a more compact form:

$$\begin{bmatrix} P_c \\ M_{\alpha,c} \\ M_{\beta,c} \end{bmatrix} = \int_0^t M_2(t-\nu)\dot{Z}(\nu)d\nu + \frac{d}{dt}\int_0^t M_3(t-\nu)\dot{Z}(\nu)d\nu + \begin{bmatrix} U[B,Z(0)] \\ 0 \\ 0 \end{bmatrix}, \qquad (3.3)$$

where

$$M_2(t) = -\frac{\rho}{m_s}\begin{bmatrix} Uc_2(t)B^* \\ \left[\frac{3}{4}c_1(t) + Uc_4(t) + aU(1-c_2(t)\right]B^* \\ -\frac{U}{2\pi}c_6(t)B^* \end{bmatrix}$$

$$M_3(t) = -\frac{\rho}{m_s}\begin{bmatrix} -c_3(t)B^* \\ \left[(\frac{a}{2}^2+\frac{1}{4})-\frac{1}{4}c_2(t)+ac_3(t)+\frac{1}{2}c_5(t)\right]B^* \\ \frac{1}{4\pi}c_7(t)B^* \end{bmatrix}.$$

Finally the equations of motion can thus be written as:

$$M_s\ddot{X}+B_s\dot{X}+K_sX = Gu+M_aX+B_aX+K_aX+\int_0^t M_2(t-\delta)\dot{Z}(\delta)d\delta$$

$$+\frac{d}{dt}\int_0^t M_3(t-\delta)\dot{Z}(\delta)d\delta+\begin{bmatrix} UB*Z(0) \\ 0 \\ 0 \end{bmatrix}\left(\frac{-\rho}{m_s}\right) \qquad (3.4)$$

,

or

$$\dot{Z}(t) = AZ(t)+Hu(t)+[B,Z(0)]\begin{bmatrix} 0 \\ 0 \\ 0 \\ v \end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ 0 \\ 0 \\ \int_0^t (M_s-M_a)^{-1}M_2(t-\delta)\dot{Z}(\delta)d\delta \end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{d}{dt}(M_s-M_a)^{-1}\int_0^t M_3(t-\delta)\dot{Z}(\delta)d\delta \end{bmatrix}, \qquad (3.5)$$

where

$$A=\left[\begin{array}{ccc:ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline \multicolumn{3}{c:}{-(M_s-M_a)^{-1}(K_s-Ka)} & \multicolumn{3}{c}{-(M_s-M_a)^{-1}(B_s-B_a)} \end{array}\right],$$

$$H=\begin{bmatrix}0\\0\\0\\(M_s-M_a)^{-1}G\end{bmatrix}\;, \qquad V=\frac{-\rho}{m_s}(M_s-M_a)^{-1}\begin{bmatrix}U\\0\\0\end{bmatrix}.$$

Since

$$(M_s-M_a)^{-1}M_2(\infty)Z(0)=[B,Z(0)]\,v$$

and

$$\int_0^t M_2(t-\nu)\dot{Z}(\nu)d\nu=M_2(\infty)Z(t)-M_2(\infty)Z(0)$$
$$+\int_0^t \tilde{M}_2(t-\nu)\dot{Z}(\nu)d\nu \quad .$$

We have

$$\dot{Z}=(A+\begin{bmatrix}0\\v\end{bmatrix}B^*)Z(t)+\begin{bmatrix}0\\0\\0\\ \int_0^t(M_s-M_a)^{-1}\tilde{M}_2(t-\nu)\dot{Z}(\nu)d\nu\end{bmatrix}$$
$$+\begin{bmatrix}0\\0\\0\\ \frac{d}{dt}\int_0^t(M_s-M_a)^{-1}M_3(t-\nu)\dot{Z}(\nu)d\nu\end{bmatrix}+Hu(t)\quad , \qquad (3.6)$$

where

$$\tilde{M}_2(t)=\begin{bmatrix} U(c_2(t)-1)B^* \\ [\frac{3}{4}c_1(t)+Uc_4(t)+aU(1-c_2(t))]B^* \\ -\frac{U}{2\pi}c_6(t)B^* \end{bmatrix}$$

and

$$M_3(\infty)=(-\frac{\rho}{m_s})\begin{bmatrix} B^* \\ (\frac{a}{2}^2-a-\frac{1}{4})B^* \\ [(4c+1)\cos^{-1}c-(2+3c)\sqrt{1-c^2}]B^* \end{bmatrix}.$$

Now we can rewrite (3.6) as

$$\dot{Z}(t)=QZ(t)+\int_0^t K_2(t-\eta)\dot{Z}(\eta)d\eta+\frac{d}{dt}\int_0^t K_3(t-\eta)\dot{Z}(\eta)d\eta$$
$$+Hu(t) \quad , \qquad (3.7)$$

where

$$K_2(\infty)=0 \quad .$$

Moreover straightforward calculation shows that

$$\begin{aligned} K_2(t) &= 0(t^{-2}) \quad \text{as } t \longrightarrow \infty \\ &= 0(t^{-\frac{1}{2}}) \quad \text{as } t \longrightarrow 0 \end{aligned} \quad .$$

The function $K_3(\cdot)$ is bounded and absolutely continuous on the interval $(0,\infty)$. Thus let $K_4(t)=K_3(t)-K_3(\infty)$ and carry out the differentiation in (3.7), we get

$$\dot{z}(t) = \tilde{A}z(t) + \int_0^t \tilde{Q}(t-\lambda)\dot{z}(\lambda)d\lambda + \tilde{H}u(t) , \qquad (3.8)$$

where

$$\tilde{Q}(t) = O(t^{-\frac{1}{2}}) \quad \text{as } t \longrightarrow 0^+$$

$$= O(t^{-2}) \quad \text{as } t \longrightarrow \infty .$$

We shall henceforth work with (3.8).

Finally in this chapter, we calculate the matrices in equation (3.8) by assuming the following section parameters:

$\omega_\alpha = 100$	rad/sec	$a = -0.4$	$x_\beta = 0.0125$
$\omega_\beta = 50$	rad/sec	$c = 0.6$	$r_\beta^2 = 0.00625$
$\omega_h = 300$	rad/sec	$x_\alpha = 0.2$	$\zeta_\beta = 0$
$\mu = 40$		$r_\alpha^2 = 0.25$	

We get

$$\tilde{H} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2243.62 \\ -43.43 \\ 18977.03 \end{bmatrix}$$

$$
\tilde{A} = \left[\begin{array}{ccc|ccc}
 & 0_{3\times 3} & & & I_{3\times 3} & \\
\hdashline
-874.45 & -1619.66 - 0.0218U^2 & 2358.04 + 0.026U^2 & -0.0154U & -0.03U & 0.066U \\
-1678.5 & 8581.9 - 0.017U^2 & 320.175 - 0.064U^2 & -0.051U & -0.149U & -0.047U \\
-10154.95 & 1591.25 - 0.263U^2 & -19189.20 - 0.015U^2 & -0.023U & -0.40U & -0.0568U
\end{array}\right]_{6\times 6}
$$

CHAPTER 4

STATE SPACE THEORY

INTRODUCTION

From the previous chapter, it was noted that the Laplace transform of function $\tilde{Q}(t)$ is not rational. Hence our integro-differential equation does not admit to a finite dimensional state space representation. A universal state space formulation is due to Balakrishnan [10]. A general infinite dimensional state space representation for our model is proposed. Then we show that the solution exists, is unique and depends continuously on the initial data. Moverover the controllability and observability issues are also discussed.

STATE SPACE REPRESENTATION

The objective of this section is to develop a state space representation for the linear system described by (3.8) using the techniques as indicated in [10].

Before a state space representation can be given for (3.8), some operators shall be defined first.

Let $H_1 = R^6$ with the usual norm; $H_2 = L_p$-space of 6x1 functions $f(\cdot)$ on $[0,\infty)$ with norm defined by

$$\|f\|_P = \left[\int_0^\infty \|f(t)\|^P_{R^6}\, dt\right]^{\frac{1}{P}} \quad \text{where } \frac{4}{3} \leqq P < 2 \quad .$$

From now on we will use the symbol $\|\cdot\|$ to denote any one of several norms when it is clear from the context which norm is intended.

Let H denote the product space $H_1 x H_2$. This will be our state space. Let A_2 denote the operator with domain $D(A_2)$ in H_2 defined by

$$A_2 f = g \quad ; \quad g(\eta) = \frac{d}{d\eta} f(\eta) \quad .$$

$$D(A_2) = \Big\{ f \epsilon H_2 \mid f(\cdot) \text{ is absolutely continuous and } f'(\cdot) \epsilon H_2 \Big\} \quad .$$

And define operator B_2, C_2 as follows:

$$B_2 : R^6 \rightarrow H_2 \qquad B_2 u = f \quad ; \qquad f(\xi) = \tilde{Q}(\xi) u$$

$$C_2 f = f(0) \quad \text{Domain of } C_2 = \Big\{ h \epsilon H_2 \mid h(\cdot) \text{ is continuous} \Big\} \quad .$$

After defining operators A_2, B_2, C_2, it is claimed that the following set of equations is a state space representation for (3.8).

$$\begin{cases} \dot{x}_1(t) = \tilde{A}x_1(t) + C_2x_2(t) + \tilde{H}u(t) \\ \dot{x}_2(t) = A_2x_2(t) + B_2\tilde{A}x_1(t) + B_2C_2x_2(t) + B_2\tilde{H}u(t) \end{cases} .$$

These can be rewritten in the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ z(t) = Cx(t) \end{cases} , \tag{4.1}$$

where

$$x(t) = [x_1(t) \quad x_2(t)]^T$$

$$A = \begin{bmatrix} \tilde{A} & C_2 \\ B_2\tilde{A} & A_2+B_2C_2 \end{bmatrix}, \quad B = \begin{bmatrix} \tilde{H} \\ B_2\tilde{H} \end{bmatrix} \text{ and } C = [I, 0] .$$

To verify this claim, see [10]. The main result of this section is

<u>Theorem 4.1.</u> The operator A defined by (4.1) generats a C_0-semigroup on H_1xH_2.

This theorem is an immediate consequence of the technical Lemmas (4.1) - (4.4) given below. The proof of Theorem 4.1 will be given after these lemmas. We will concentrate on the "degenerate" A (i.e., $B_2 = 0$) first.

<u>Lemma 4.1.</u> Given the "degenerate" operator

$$A = \begin{bmatrix} \tilde{A} & C_2 \\ 0 & A_2 \end{bmatrix}$$

with $\widetilde{A}$, C_2, and A_2 as defined above, then A is a linear operator with dense domain.

Proof. Straightforward.

Lemma 4.2. If $x(\cdot)\in D(A_2)$, then $\|x(0)\| \leqq 2(\|x\|_p + \|x'\|_p)$,

$$\frac{4}{3} \leqq p < 2 \quad .$$

Remark: This lemma remains valid for more general p.

Proof. First consider the restriction of $x(\cdot)$ on $[0,1]$.

$$x(t) = x(0) + \int_0^t x'(t)\, dt \qquad \forall\, t\ [0,1] \quad .$$

or

$$x(0) = x(t) - \int_0^t x'(t)\, dt$$

Hence

$$\|x(o)\| \leqq \|x(t)\| + \int_0^t \|x'(t)\|\ dt$$

$$\leqq \|x(t)\| + \|x'\|_p\, t^{\frac{1}{q}} \qquad t \in [0,1]$$

$$\leqq \|x(t)\| + \|x'\|_p$$

$$\|x(0)\|^p \leqq 2^p(\|x(t)\|^p + \|x'\|_p^p) \quad .$$

Integrate in t,

$$\int_0^1 \|x(0)\|^p \, dt \leq 2^p \left\{ \int_0^1 \|x(t)\|^p \, dt + \|x'\|_p^p \right\} .$$

Hence

$$\|x(0)\|^p \leq 2^p \left(\|x\|_p^p + \|x'\|_p^p \right)$$

$$\|x(0)\| \leq 2 \left(\|x\|_p + \|x'\|_p \right) .$$

Q.E.D.

<u>Lemma 4.3.</u>

$$A = \begin{bmatrix} \tilde{A} & C_2 \\ 0 & A_2 \end{bmatrix}$$

is a closed linear operator on H.

<u>Proof.</u> Given that

$$\left\{ \begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} \right\} \subseteq D(A) \equiv \dot{R}^6 \times D(A_2) ,$$

$$\begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} \to \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} \to \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} ,$$

need to show that

$$\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \in D(A) \quad \text{and} \quad A \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} .$$

It is shown in Balakrishnan [12] that $x^2 \in D(A_2)$ and $A_2 x^2 = y^2$. We only need to show that $\tilde{A}x^1 + C_2 x^2 = y^1$. According to Lemma 4.2,

$$\|C_2 x_n^2 - C_2 x^2\| = \|x_n^2(0) - x^2(0)\|$$

$$\leqq \|x_n^2 - x^2\|_p + \|x'^2_n - x'^2\|$$

$$\rightarrow 0$$

by assumption and the result above. Hence $\tilde{A}x_n^1 + C_2 x_n^2$ converges to $\tilde{A}x^1 + C_2 x^2$ and $y^1 = \tilde{A}x^1 + C_2 x^2$.

Q.E.D.

Now we are ready to show

Lemma 4.4.

$$A = \begin{bmatrix} \tilde{A} & C_2 \\ 0 & A_2 \end{bmatrix}$$

generates a C_0-semigroup on $H \equiv R^6 x L_p[0,\infty)^6$ with $\frac{4}{3} \leqq p < 2$.

Remark. The linear operator

$$\begin{bmatrix} \tilde{A} & 0 \\ 0 & A_2 \end{bmatrix}$$

is an infinitesimal generator of a contraction C_0-semigroup. C_2 is an unbounded, unclosable operator by itself. This lemma indicates that

$$\begin{bmatrix} \tilde{A} & 0 \\ 0 & A_2 \end{bmatrix}$$

will generate a semigroup under the perturbation of C_2.

<u>Proof.</u> From Lemma 4.1., 4.3., A is linear closed and has a dense domain. Moreover, it is easy to see that

$$\|R(\lambda;\tilde{A})\| \leqq \frac{1}{\lambda-\|\tilde{A}\|} \qquad \forall\ \lambda > \|\tilde{A}\|$$

$$\|R(\lambda;A_2)\| \leqq \frac{1}{\lambda} \qquad \forall\ \lambda > 0 \qquad .$$

Set

$$R = \begin{bmatrix} R(\lambda;\tilde{A}) & R(\lambda;\tilde{A})C_2R(\lambda;A_2) \\ 0 & R(\lambda;A_2) \end{bmatrix},$$

then

$$(\lambda I-A)R = \left\{\lambda I - \begin{bmatrix} \tilde{A} & 0 \\ 0 & A_2 \end{bmatrix} - \begin{bmatrix} 0 & C_2 \\ 0 & 0 \end{bmatrix}\right\} R$$

$$= \left\{\lambda I - \begin{bmatrix} \tilde{A} & 0 \\ 0 & A_2 \end{bmatrix}\right\} R - \begin{bmatrix} 0 & C_2 \\ 0 & 0 \end{bmatrix} R$$

$$= \begin{bmatrix} \lambda I-\tilde{A} & 0 \\ 0 & \lambda I-A_2 \end{bmatrix} R - \begin{bmatrix} 0 & C_2R(\lambda;A_2) \\ 0 & 0 \end{bmatrix}$$

$$= I$$

and for every $x \in D(A)$,

$$R(\lambda I-A)x = R\left\{\lambda I - \begin{bmatrix} \tilde{A} & 0 \\ 0 & A_2 \end{bmatrix} - \begin{bmatrix} 0 & C_2 \\ 0 & 0 \end{bmatrix}\right\} x$$

$$= R\left\{\lambda I - \begin{bmatrix} \tilde{A} & 0 \\ 0 & A_2 \end{bmatrix}\right\}x - R\begin{bmatrix} 0 & C_2 \\ 0 & 0 \end{bmatrix}x$$

$$= \begin{bmatrix} I & R(\lambda;\tilde{A})C_2 \\ 0 & I \end{bmatrix}x - \begin{bmatrix} 0 & R(\lambda;\tilde{A})C_2 \\ 0 & 0 \end{bmatrix}x$$

$$= x \quad .$$

Therefore the resolvent of

$$\begin{bmatrix} \tilde{A} & C_2 \\ 0 & A_2 \end{bmatrix}$$

exists for $\lambda > \|\tilde{A}\|$ and it is given by the operator R.

Next, we contend that there exist some constants $M > 0$ and ω such that

$$\|R(\lambda;A)^n\| \leq \frac{M}{(\lambda-\omega)^n} \quad \text{for} \quad \lambda > \omega \;, \qquad n = 1, 2, \ldots .$$

For $n = 1$,

$$\left\| R \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|^2 = \|R(\lambda;\tilde{A})v_1 + R(\lambda;\tilde{A})C_2R(\lambda;A_2)v_2\|^2 + \|R(\lambda;A_2)v_2\|^2$$

$$\leq 2\left\{ \frac{\|v_1\|^2}{(\lambda-\|\tilde{A}\|)^2} + \|R(\lambda;\tilde{A})\|^2 \, \|C_2R(\lambda;A_2) \cdot v_2\|^2 \right\} + \frac{\|v_2\|^2}{\lambda^2}$$

$$\forall \lambda > \|\tilde{A}\| \quad .$$

Note here that

$$\|C_2R(\lambda;A_2)v_2\| = \left\|C_2 \int_0^\infty e^{-\lambda t}\, v_2(t+\cdot)\, dt\right\|$$

$$= \left\|\int_0^\infty e^{-\lambda t}\, v_2(t)\, dt\right\|$$

$$\leqq \|v_2\|_p \left(\int_0^\infty e^{-\lambda q t}\, dt\right)^{\frac{1}{q}}$$

$$= \|v_2\|_p \left(\frac{1}{\lambda q}\right)^{\frac{1}{q}} \quad \text{where } \tfrac{4}{3} \leqq p < 2$$

$$\text{and } \frac{1}{p} + \frac{1}{q} = 1 \quad .$$

Hence

$$\left\| R \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|^2 \leqq \frac{2\|v_1\|^2}{(\lambda - \|\tilde{A}\|)^2} + \|v_2\|^2 \left\{ \frac{2 \left(\frac{1}{\lambda q}\right)^{2/q}}{(\lambda - \|\tilde{A}\|)^2} + \frac{1}{\lambda^2} \right\} \quad .$$

For $\frac{4}{3} \leqq p < 2$, we have the following extimates:

a) $2\left[\frac{1}{q}\right]^{\frac{2}{q}} \leqq 1$

b) $\left[\frac{1}{\lambda}\right]^{\frac{2}{q}} < 1 \quad \forall \lambda > \|\tilde{A}\|$ ($\gg 1$ for normal speed of aircraft).

So

$$\left\| R \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|^2 \leqq \frac{2}{(\lambda - \|\tilde{A}\|)^2} \left[\|v_1\|^2 + \|v_2\|^2 \right] = \frac{2}{(\lambda - \|\tilde{A}\|)^2}$$

$$\cdot \left\| \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|^2 ,$$

i.e.,

$$\|R\| \leqq \frac{2}{\lambda - \|\tilde{A}\|} \qquad \forall \ \lambda > \|\tilde{A}\| \quad .$$

For $n = 2$,

$$\left\| R^2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|^2 \leqq \frac{2\|v_1\|^2}{(\lambda - \|\tilde{A}\|)^4} + \|v_2\|^2 \left\{ \frac{2\left[\frac{1}{q\lambda}\right]^{\frac{2}{q}} \left[\frac{1}{\lambda - \|\tilde{A}\|} + \frac{1}{\lambda}\right]^2}{(\lambda - \|\tilde{A}\|)^2} + \frac{1}{\lambda^4} \right\}$$

$$\leqq \frac{2\|v_1\|^2}{(\lambda - \|\tilde{A}\|)^4} + \frac{\|v_2\|^2}{(\lambda - \|\tilde{A}\|)} \left\{ \frac{2(\lambda - \|\tilde{A}\|) + \|\tilde{A}\|^2}{\lambda^{2+\frac{2}{q}}} + \left\{ 1 - \frac{\|\tilde{A}\|}{\lambda} \right\}^4 \right\}$$

$$\leqq \frac{2}{(\lambda - \|\tilde{A}\|)^4} \left\| \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|^2$$

$$\|R^2\| \leqq \frac{\sqrt{2}}{(\lambda - \|\tilde{A}\|)^2} \qquad \forall \ \lambda > \|\tilde{A}\| \quad .$$

Proceeding inductively we find for general n,

$$\left\| R^n \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|^2 \leqq \frac{2\|v_1\|^2}{(\lambda-\|\tilde{A}\|)^{2n}} + \frac{\|v_2\|^2}{(\lambda-\|\tilde{A}\|)^{2n}} \left[\left(1 - \frac{\|\tilde{A}\|}{\lambda}\right)^{2n} \right.$$

$$\left. + \frac{(n\lambda^{n-2}(\lambda-\|\tilde{A}\|)+\|\tilde{A}\|^{n-1})^2}{\lambda^{2(n-1)+\frac{2}{q}}} \right] \leqq \frac{2}{(\lambda-\|\tilde{A}\|)^{2n}} \left\| \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|^2 .$$

Hence

$$\|R^n\| \leqq \frac{\sqrt{2}}{(\lambda-\|\tilde{A}\|)^n} \quad \text{for} \quad \lambda > \|\tilde{A}\| \ , \ n=1,2,\ldots$$

It follows that A is the infinitesimal generator of a C_0-semigroup $S(t)$, satisfying $|S(t)| \leqq \sqrt{2}\, e^{\|\tilde{A}\| t}$.

Q.E.D.

Lemma 4.5. $\hat{A} \equiv \begin{bmatrix} \tilde{A} & C_2 \\ 0 & A_2+B_2C_2 \end{bmatrix}$ also generates a C_0-semigroup on $H \equiv R^6 \times L_p(0,\infty)^6$ with $\frac{4}{3} \leqq p < 2$.

Proof: First consider $\left\| \begin{bmatrix} 0 & 0 \\ 0 & B_2C_2R(\lambda;A_2) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|$

$$= \|B_2C_2R(\lambda;A_2)v_2\| \leqq \|B_2\| \frac{1}{(\lambda q)^{\frac{1}{q}}} \|v_2\| .$$

Hence

$$\frac{\|B_2\|}{(\lambda q)^{\frac{1}{q}}} < 1 \quad \text{provided} \quad \|B_2\| < (\lambda q)^{\frac{1}{q}} \quad \text{or}$$

$$\|B_2\|^q < \lambda q \quad .$$

Next set

$$R = \sum_{k=0}^{\infty} R(\lambda;A)\left(\begin{bmatrix} 0 & 0 \\ 0 & B_2C_2 \end{bmatrix} R(\lambda;A)\right)^k$$

$$= R(\lambda;A)\left(I - \begin{bmatrix} 0 & 0 \\ 0 & B_2C_2 \end{bmatrix} R(\lambda;A)\right)^{-1}$$

provided $\lambda > \frac{\|B_2\|^q}{q}$, $2 < q \leqq 4$.

We contend that R is the resolvent of

$$\begin{bmatrix} \tilde{A} & C_2 \\ 0 & A_2+B_2\ C_2 \end{bmatrix} ;$$

$$(\lambda I-\hat{A})R = \left\{\lambda I - \begin{bmatrix} \tilde{A} & C_2 \\ 0 & A_2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & B_2C_2 \end{bmatrix}\right\} R$$

$$= \left\{\lambda I - \begin{bmatrix} \tilde{A} & C_2 \\ 0 & A_2 \end{bmatrix}\right\} R - \begin{bmatrix} 0 & 0 \\ 0 & B_2C_2 \end{bmatrix} R$$

$$= \begin{bmatrix} \lambda I-\tilde{A} & -C_2 \\ 0 & \lambda I-A_2 \end{bmatrix} R - \begin{bmatrix} 0 & 0 \\ 0 & B_2C_2 \end{bmatrix} R$$

$$= \begin{bmatrix} \lambda I-\tilde{A} & -C_2 \\ 0 & \lambda I-A_2 \end{bmatrix} \begin{bmatrix} R(\lambda;\tilde{A}) & R(\lambda;\tilde{A})C_2R(\lambda;A_2) \\ 0 & R(\lambda;A_2) \end{bmatrix}$$

$$\left(I - \begin{bmatrix} 0 & 0 \\ 0 & B_2C_2R(\lambda;A_2) \end{bmatrix}\right)^{-1} - \begin{bmatrix} 0 & 0 \\ 0 & B_2C_2R(\lambda;A_2) \end{bmatrix} \cdot$$

$$\left(I - \begin{bmatrix} 0 & 0 \\ 0 & B_2C_2R(\lambda;A_2) \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\left(I - \begin{bmatrix} 0 & 0 \\ 0 & B_2C_2R(\lambda;A_2) \end{bmatrix}\right)^{-1}$$

$$- \begin{bmatrix} 0 & 0 \\ 0 & B_2C_2R(\lambda;A_2) \end{bmatrix}\left(I - \begin{bmatrix} 0 & 0 \\ 0 & B_2C_2R(\lambda;A_2) \end{bmatrix}\right)^{-1}$$

$$= \quad I$$

and for every $x \in D(A)$,

$$R(\lambda I-\hat{A})x = R\left\{\lambda I - \begin{bmatrix} \tilde{A} & C_2 \\ 0 & A_2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & B_2C_2 \end{bmatrix}\right\}x$$

$$= R\left\{\lambda I - \begin{bmatrix} \tilde{A} & C_2 \\ 0 & A_2 \end{bmatrix}\right\}x - R\begin{bmatrix} 0 & 0 \\ 0 & B_2C_2 \end{bmatrix}x$$

$$= \sum_{k\ 0}^{\infty} R(\lambda;A\ )(\begin{bmatrix} 0 & 0 \\ 0 & B_2C_2 \end{bmatrix}R(\lambda;A)\ )^k \left\{\lambda I - \begin{bmatrix} \tilde{A} & C_2 \\ 0 & A_2 \end{bmatrix}\right\}x$$

$$- \sum_{k=0}^{\infty} R(\lambda;A)\left(\begin{bmatrix} 0 & 0 \\ 0 & B_2C_2 \end{bmatrix} R(\lambda;A)\right)^k \begin{bmatrix} 0 & 0 \\ 0 & B_2C_2 \end{bmatrix} x$$

$$= \sum_{k=0}^{\infty}\left(R(\lambda;A)\begin{bmatrix} 0 & 0 \\ 0 & B_2C_2 \end{bmatrix}\right)^k x - \sum_{k=0}^{\infty}\left(R(\lambda;A)\begin{bmatrix} 0 & 0 \\ 0 & B_2C_2 \end{bmatrix}\right)^{k+1} x$$

$$= x + \sum_{k=1}^{\infty}\left(R(\lambda;A)\begin{bmatrix} 0 & 0 \\ 0 & B_2C_2 \end{bmatrix}\right)^k x$$

$$- \sum_{k=0}^{\infty}\left(R(\lambda;A)\begin{bmatrix} 0 & 0 \\ 0 & B_2C_2 \end{bmatrix}\right)^{k+1} x$$

$$= x \quad .$$

Therefore the resolvent of

$$\begin{bmatrix} \tilde{A} & C_2 \\ 0 & A_2+B_2C_2 \end{bmatrix}$$

exists for

$$\lambda > \|\tilde{A}\| \quad \text{and} \quad \lambda > \frac{\|B_2\|^q}{q} \quad ; \quad 2 < q \leq 4 \quad .$$

And it is given by the operator R.

Finally we have to show that there exist some constants $M > 0$ and ω such that

$$\| R(\lambda; \hat{A}) \| \leq \frac{M}{(\lambda - \omega)^n} \qquad \text{for } \lambda > \omega \;, \quad n = 1,2,3,\ldots$$

Recall that in Lemma 4.4. $\begin{bmatrix} \tilde{A} & C_2 \\ 0 & A_2 \end{bmatrix}$ generates a C_0-semigroup $S(t)$ on H with $\frac{4}{3} \leq p < 2$ and $\|S(t)\| \leq \sqrt{2}\, e^{\|\tilde{A}\| t}$.

For $n=1$,

$$\left\| R \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\|^2 = \left\| R(\lambda;A)\left(I - \begin{bmatrix} 0 & 0 \\ 0 & B_2C_2 \end{bmatrix} R(\lambda;A)\right)^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\|^2$$

$$\left\| \begin{bmatrix} R(\lambda;\tilde{A}) & R(\lambda;\tilde{A})C_2R(\lambda;A_2) \\ 0 & R(\lambda;A_2) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (I-B_2C_2R(\lambda;A_2)^{-1} \end{bmatrix} \right.$$

$$\left. \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\|^2 \qquad \ldots(*)$$

Multiplying it out, we get

$$(*) = \left\| \begin{matrix} R(\lambda;\tilde{A})u_1 + R(\lambda;\tilde{A})C_2R(\lambda;A_2)(I-B_2C_2R(\lambda;A_2))^{-1}u_2 \\ R(\lambda;A_2)(I-B_2C_2R(\lambda;A_2))^{-1}u_2 \end{matrix} \right\|^2$$

$$\leq \quad 2(\|R(\lambda;\tilde{A})u_1\|^2 + \|R(\lambda;\tilde{A})C_2R(\lambda;A_2)(I-B_2C_2R(\lambda;A_2))^{-1}$$

$$\cdot u_2\|^2) + \|R(\lambda;A_2)(I-B_2C_2R(\lambda;A_2))^{-1}u_2\|^2$$

$$\leq 2\left(\frac{1}{(\lambda-\|\tilde{A}\|)^2}\|u_1\|^2 + \frac{1}{(\lambda-\|\tilde{A}\|)^2}\left(\frac{1}{(q\lambda)^{1/q}}\,\frac{1}{1-\frac{\|B_2\|}{(q\lambda)^{1/q}}}\right)^2 \|u_2\|^2\right) + \frac{1}{\lambda^2}\left(\frac{1}{1-\frac{\|B_2\|}{(q\lambda)^{1/q}}}\right)^2 \|u_2\|^2$$

$$\leq \frac{2}{(\lambda-\|\tilde{A}\|)^2}\|u_1\|^2 + \frac{2}{(\lambda-\|\tilde{A}\|)^2}\left(\frac{1}{(q\lambda)^{1/q}-\|B_2\|}\right)^2 \|u_2\|^2 + \frac{1}{\lambda^2}\left(\frac{1}{1-\frac{\|B_2\|}{(q\lambda)^{1/q}}}\right)^2 \|u_2\|^2 .$$

Note that

$$\frac{1}{\left(\lambda-\frac{\|B_2\|\lambda}{(q\lambda)^{1/q}}\right)^2} = \left(\frac{1}{\lambda-\frac{\|B_2\|}{(q)^{1/q}}\lambda^{1-1/q}}\right)^2$$

$$\leq \left(\frac{1}{\lambda-\frac{\|B_2\|}{\sqrt{2}}\lambda^{1-\frac{1}{q}}}\right)^2$$ Recall that $(\frac{1}{q})^{1/q} \leq \frac{1}{2}$ for $2 < q \leq 4$

$$\leq \left(\frac{1}{\lambda-\frac{\|B_2\|}{\sqrt{2}}\lambda^{3/4}}\right)^2 \leq \left(\frac{1}{\lambda-\frac{\|B_2\|}{\sqrt{2}}\left(\frac{1+3\lambda}{4}\right)}\right)^2$$

$$\leq \frac{1}{\left(\lambda\left(1-\frac{3\|B_2\|}{4\sqrt{2}}\right)-\frac{\|B_2\|}{4\sqrt{2}}\right)^2} \qquad \left(\text{for } \|B_2\| < \frac{8\sqrt{2}}{3}\right)$$

$$\leq \frac{\dfrac{1}{\dfrac{3\|B_2\|}{4\sqrt{2}} - 1}}{\left(\lambda + \dfrac{\|B_2\|}{3\|B_2\| - 4\sqrt{2}}\right)^2}$$

where

$$\frac{1}{\dfrac{3\|B_2\|}{4\sqrt{2}} - 1} < 1$$

Hence

$$\left\| R \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\|^2 \leq \frac{2}{(\lambda - \|\tilde{A}\|)^2} \|u_1\|^2 + \frac{2}{(\lambda - \|\tilde{A}\|)^2} \|u_2\|^2$$

i.e.

$$\|R\| \leq \frac{\sqrt{2}}{\lambda - \|\tilde{A}\|} \qquad \forall\ \lambda > \|A\| \ \text{ and } \lambda > \frac{\|B_2\|^q}{q}$$

or

$$\|R\| \leq \frac{\sqrt{2}}{\lambda - \|\tilde{A}\| - \dfrac{\|B_2\|^q}{q}} \quad \forall\ \lambda > \|\tilde{A}\| + \frac{\|B_2\|^q}{q}$$

For $n = 2$,

$$\left\| R^2 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} R^2(\lambda;\tilde{A}) & \Delta \\ 0 & R^2(\lambda;A_2)[I - B_2C_2R(\lambda;A_2)]^{-2} \end{bmatrix} \right\|.$$

$$\cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Bigg\|^2$$

where

$$\Delta \equiv R(\lambda;\tilde{A})C_2R(\lambda;A_2)\left(\left(I-B_2C_2R(\lambda;A_2)\right)^{-1}\right)^2 \quad .$$

Hence

$$\left\| R^2 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} R^2(\lambda;\tilde{A})u_1 + \Delta u_2 \\ R^2(\lambda;A_2)\left[I-B_2C_2R(\lambda;A_2)\right]^{-2}u_2 \end{bmatrix} \right\|^2$$

$$\leq \quad 2\left[\|R^2(\lambda;\tilde{A})u_1\|^2 + \|\Delta u_2\|^2\right] + \|R^2(\lambda;A_2)\left[I-B_2C_2\right.$$

$$\left.R(\lambda;A_2)\right]^{-2}u_2\|^2$$

$$\leq \quad 2\left[\frac{\|u_1\|^2}{(\lambda-\|\tilde{A}\|)^4} + \|u_2\|^2\left(\frac{1}{\lambda-\|\tilde{A}\|}\,\frac{1}{(q\lambda)^{1/q}}\,\frac{1}{1-\frac{\|B_2\|}{(q\lambda)^{1/q}}}\right)^4\right]$$

$$+ \frac{\|u_2\|^2}{\lambda^4}\,\frac{1}{\left(1-\frac{\|B_2\|}{(q\lambda)^{1/q}}\right)^4}$$

$$\leq \quad \frac{2\|u_1\|^2}{(\lambda-\|\tilde{A}\|)^4} + \frac{2\|u_2\|^2}{(\lambda-\|\tilde{A}\|)^4(q\lambda)^{4/q}\left(1-\frac{\|B_2\|}{(q\lambda)^{1/q}}\right)^4}$$

$$+\frac{\|u_2\|^2}{\lambda^4}\frac{1}{\left(1-\frac{\|B_2\|}{(q\lambda)1/q}\right)^4}$$

$$\leq \frac{2\|u_1\|^2}{(\lambda-\|\widehat{A}\|)^4}+\frac{\|u_2\|^2}{\left(1-\frac{\|B_2\|}{(q\lambda)^{1/q}}\right)^4(\lambda-\|\widetilde{A}\|)^4}\left[\left\{\frac{1}{(q\lambda)^{1/q}}\right\}^4\right.$$

$$\left.+\left(1-\frac{\|\widetilde{A}\|}{\lambda}\right)^4\right]$$

But

$$\frac{\left[\left(1-\frac{\|A\|}{\lambda}\right)^4+\left(\frac{1}{(q\lambda)^{1/q}}\right)^4\right](q\lambda)^{4/q}}{\left((q\lambda)^{1/q}-\|B_2\|\right)^4}\leq 2$$

$$\forall\ \lambda>\|\widetilde{A}\| \text{ and } \lambda>\frac{\|B_2\|^q}{q}$$

$$\|R^2\|\leq\frac{\sqrt{2}}{(\lambda-\|\widetilde{A}\|)^2}\leq\frac{\sqrt{2}}{\left(\lambda-\|\widetilde{A}\|-\frac{\|B_2\|^q}{q}\right)^2}$$

$$\forall\ \lambda>\|\widetilde{A}\|+\frac{\|B_2\|^q}{q}$$

Proceeding inductively we find for general n,

$$\left\| R^n \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\|^2 \leqq \frac{2\|u_1\|^2}{(\lambda - \|\tilde{A}\|)^{2n}} + \frac{\|u_2\|^2}{(\lambda - \|\tilde{A}\|)^{2n}}$$

$$\cdot \left[\frac{(\lambda q)^{2n/q} \left\{ (1 - \frac{\|\tilde{A}\|}{\lambda})^{2n} + \left\{ \frac{1}{(q\lambda)^{1/q}} \right\}^{2n} \right\}}{((\lambda q)^{1/q} - \|B_2\|)^{2n}} \right].$$

Hence

$$\|R^n\| \leqq \frac{\sqrt{2}}{(\lambda - \|\tilde{A}\| - \frac{\|B_2\|^q}{q})^n}, \quad \forall \ \lambda > \|\tilde{A}\| + \frac{\|B_2\|^q}{q}, \quad n = 1, 2, \ldots$$

It follows that $\begin{bmatrix} \tilde{A} & C_2 \\ 0 & A_2 + B_2 C_2 \end{bmatrix}$ is the infinitesimal generator of a C_0-semigroup. ¶

We are now able to prove our main theorem.

<u>Proof of Theorem 4.1.</u>

From previous lemmas, we have shown that $\begin{bmatrix} A & C_2 \\ 0 & A_2 + B_2 C_2 \end{bmatrix}$ is an infinitesimal generator of a C_0-semigroup $T(t)$, satisfying $\|T(t)\| \leqq \sqrt{2}\, e^{(\|\tilde{A}\| + \frac{\|B_2\|^q}{q})t}$

Observe that $\begin{bmatrix} 0 & 0 \\ B_2 \tilde{A} & 0 \end{bmatrix}$ is a bounded linear operator on the reflexive Banach space H, hence $\begin{bmatrix} \tilde{A} & C_2 \\ B_2 \tilde{A} & A_2 + B_2 C_2 \end{bmatrix}$ is the inifinitesimal generator of a C_0-semigroup $S(t)$ on H, satisfying $\|S(t)\| \leqq \sqrt{2}\, e^{\{\|\tilde{A}\| + \frac{\|B_2\|^q}{q} + \sqrt{2}\,\|B_2\|\,\|\hat{A}\|\}\, t}$,

see Pazy [11] . Q.E.D.

CHAPTER 5

ACTIVE CONTROL OF FLUTTER PROBLEM

INTRODUCTION

The vital role of flutter control problem is played by the choice of aerodynamic model. Most of the studies up to now were conducted by approximating the aerodynamic modelling with rational transfer functions. As a result, the final model obtained is described by linear, constant-coefficient, ordinary differential equations. The major problems with those approaches are as follows:

(i) The singularity near origin of the aerodynamic model will not be seen after rational approximation. This implies that the arbitrary transient response predicted will be less acurate.

(ii) The control law obtained via finite dimensional approximations is not put back into the original system. Hence the control may not be able to stabilize the original system even though it stabilized the approximating system. This can be seen from the following example.

AN EXAMPLE

Consider the following one-dimensional version of equation (3.8):

$$\dot{z}(t) = z(t) + \int_0^t \tilde{Q}(t-\theta) z(\theta) d\theta + u(t) \tag{5.1}$$

where $\tilde{Q}(t)$ in our aeroelastic system is of $O(t^{-\frac{1}{2}})$ for small t. Recall that the rational approximation of Küssner function is $\tilde{Q}(t) = 1 - 0.500e^{-0.130t} - 0.500e^{-t}$ in engineering literature.

$$\frac{d\tilde{Q}(t)}{dt} = 0.065e^{-0.130t} + \frac{1}{2}e^{-t}$$

For this one dimensional example, we are going to illustrate that if we take $\tilde{Q}(t) \approx e^{-t}$ as our corresponding rational approximation, then the feedback control law derived from approximate model will not be able to stabilize the original system.

First we rewrite the equation (5.1) as follows:

$$\dot{z}(t) = z(t) + w(t) + u(t) \tag{5.2}$$

where

$$w(t) = \int_0^t e^{-(t-\theta)} \dot{z}(\theta) d\theta \tag{5.3}$$

Differentiating (5.3),

$$\begin{aligned} \dot{w}(t) &= \dot{z}(t) - w(t) \\ &= z(t) + w(t) + u(t) - w(t) \\ &= z(t) + u(t) \end{aligned}$$

Hence (5.1) can be written as:

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ z(t) &= Cx(t) \end{aligned}\right| \qquad (5.4)$$

where

$$x(t) = \begin{bmatrix} z(t) \\ w(t) \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} .$$

It is easy to check that the matrix A has one stable eigenvalue

$$\lambda_1 = \frac{1-\sqrt{5}}{2}$$

and one unstable eigenvalue

$$\lambda_2 = \frac{1+\sqrt{5}}{2} .$$

Moreover the system (5.4) is controllable and observable, it is well known that we can find a feedback control $u(t) = -kx(t)$ such that the system will be stabilized. if we take

$$u(t) = -\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1.1 & 0 \\ 0 & .2 \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} = -1.1z(t) - .2w(t)$$

then

$$BB^{*}P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1.1 & 0 \\ 0 & .2 \end{bmatrix} \cdot \begin{bmatrix} 1.1 & .2 \\ 1.1 & .2 \end{bmatrix}$$

$$A - BB^{*}P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1.1 & .2 \\ 1.1 & .2 \end{bmatrix}$$

$$= \begin{bmatrix} -.1 & .8 \\ -.1 & -.2 \end{bmatrix}$$

$$\det(A - BB^{*}P) = (\lambda + 0.1)(\lambda + .2) + .08$$

$$= \lambda^{2} + .3\lambda + .1 \Rightarrow \quad \lambda = \frac{-.3 \pm \sqrt{.09 - .4}}{2}.$$

This matrix has two stable eigenvalues, hence the feedback system is stable for an approximating model.

Now let's consider the original system (5.1) with $\tilde{Q}(t) = e_1(t)$,

$$\dot{z}(t) = z(t) + \int_0^t \tilde{Q}(t-\theta)\dot{z}(\theta)d\theta + u(t)$$

$$= z(t) + \int_0^t \tilde{Q}(t-\theta)\dot{z}(\theta)d\theta - 1.1z(t) - 0.2w(t)$$

$$= -0.1z(t) + 0.8\int_0^t c_1(t-s)\dot{z}(s)\,ds$$

Take Laplace transform of equation (5.5).

$$sZ(s) = -0.1Z(s) + 0.8\left[sZ(s) - z(0)\right]\frac{1}{s}\,\frac{Ue^{-\frac{s}{U}}}{K_o(\frac{s}{U})+K_1(\frac{s}{U})} + z(0)$$

$$Z(s) = z(0)\left(\frac{K_o(\frac{s}{U})+K_1(\frac{s}{U})-\frac{0.8}{s}Ue^{-\frac{s}{U}}}{(s+0.1)(K_o(\frac{s}{U})+K_1(\frac{s}{U}))-0.8Ue^{-\frac{s}{U}}}\right) = z(0)H(s)$$

In order to examine the transient behavior of the system, we take the asymptotic expansions for large arguments s of the Modified Bessel functions $K_o(\frac{s}{U})$ and $K_1(\frac{s}{U})$.

$$K_\nu(\tfrac{s}{U}) \sim \sqrt{\frac{\pi}{2\frac{s}{U}}}\, e^{-\frac{s}{U}}\left\{1 + \frac{\mu-1}{8\frac{s}{U}} + \ldots\right\} \text{ where } \mu = 4\nu^2 \quad .$$

Hence, $K_o(\frac{s}{U})+K_1(\frac{s}{U}) \simeq \sqrt{2\pi U}\,\frac{1}{\sqrt{s}}\,e^{-\frac{s}{U}}$ for large s.

The denominator of the transfer function H(s) now takes the following form:

$$\sqrt{\frac{2\pi U}{s}}\,(s+0.1) - 0.8U \quad .$$

Hence

$$\sqrt{\frac{2\pi U}{s}(s+0.1)} - 0.8U = 0 \text{ implies that}$$

$$\sqrt{2\pi U}(s+0.1) - 0.8U\sqrt{s} = 0 \quad .$$

Let $y = \sqrt{s}$, then

$$y = \frac{0.8U \pm \sqrt{0.64U^2 - 0.4(\sqrt{2\pi U})^2}}{2\ 2\ U} > 0 \quad \text{for flutter speed}$$

$$s = \frac{0.64U^2 + 0.64U^2 - 0.4(2\pi U) \pm 1.6U\sqrt{0.64U^2 - (0.4)2\pi U}}{8\pi U} > 0 \ .$$

The original feedback system is clearly unstable. This example illustrates that the feedback control can stabilize the approximate system but not the original system.

INPUT-OUTPUT STABILITY

In order to stabilize the original system, we will work with the following problem:

Given that

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ z(t) = Cx(t) \end{cases} \tag{5.6}$$

where

$$A=\begin{bmatrix}\tilde{A} & C_2 \\ B_2\tilde{A} & B_2C_2+A_2\end{bmatrix}, \quad B=\begin{bmatrix}\tilde{H} \\ B_2\tilde{H}\end{bmatrix} \text{ and } C=\begin{bmatrix} I & 0\end{bmatrix} ;$$

with

$$J[u]=\int_0^\infty \|x_1(t)\|^p \, dt + \int_0^\infty |u(t)|^p \, dt$$

we want to find a feedback control such that the cost functional will be finite. Note that the $x_1(t)$ is only the finite dimensional portion of the state space.

We need the following technical lemma in the proof of main theorem 5.1..

__Lemma 5.1.__ (Paley-Wiener) Let A be a matrix with entries consisting of positively supported delta distributions and L_1-functions, then it is regular if and only if its Laplace transform satisfying the following condition:

$$\inf_{s\in C^+} |\det L(A)| > 0 \quad , \text{ where } C^+ = \{ s : \mathrm{Re}\, s \geq 0 \}$$

__Proof:__ See Hille-Phillips [17].

__Theorem 5.1.__ Given that $(\hat{A},\tilde{H})$ is controllable, then there exists a feedback control u(t) such that $J[u] < \infty$.

__Proof:__ From chapter 3, it is clear that $(\tilde{A},\tilde{H})$ is controllable, then the finite dimensional Ricatti equation $\tilde{A}^*P_1+P_1\tilde{A}+I-P_1\tilde{H}\tilde{H}^*P_1=0$ has a nonnegative definite, self-adjoint P_1, such that $\tilde{A}-\tilde{H}\tilde{H}^*P_1$ is stable if $u(t)=-\tilde{H}^*P_1\cdot x_1(t)$.

In order to show that J[u] is finite, we need the resolvent for $\begin{bmatrix} \tilde{A} & C_2 \\ B_2\tilde{A} & B_2C_2+A_2 \end{bmatrix}$ first.

Recall that the resolvent for $\begin{bmatrix} \tilde{A} & C_2 \\ 0 & B_2C_2+A_2 \end{bmatrix}$ is

$$\hat{R}=\begin{bmatrix} R(\lambda;\tilde{A}) & R(\lambda;\tilde{A})C_2R(\lambda;A_2)\left\{I-B_2C_2R(\lambda;A_2)\right\}^{-1} \\ 0 & R(\lambda;\tilde{A})\left\{I-B_2C_2R(\lambda;A_2)\right\}^{-1} \end{bmatrix}$$

for $s>\|\tilde{A}\|+\dfrac{\|B_2\|^q}{q}$

Define $R_{11}\equiv R(\lambda;\tilde{A})$

$$R_{12}\equiv R(\lambda;\tilde{A})C_2R(\lambda;A_2)\left\{I-B_2C_2R(\lambda;A_2)\right\}^{-1}$$

$$R_{22}\equiv R(\lambda;A_2)\left\{I-B_2C_2R(\lambda;A_2)\right\}^{-1} \quad .$$

Then $A=\begin{bmatrix} \tilde{A} & C_2 \\ B_2\tilde{A} & B_2C_2+A_2 \end{bmatrix}$ will have the resolvent

$$R = \hat{R}\sum_{k=0}^{\infty}\begin{bmatrix}0 & 0\\ B_2\hat{A} & 0\end{bmatrix}\hat{R}^{\,k} \qquad \text{for } \lambda > \|B_2\|\cdot\|\tilde{A}\| + \|\tilde{A}\| + \frac{\|B_2\|^q}{q} \quad .$$

$$= \hat{R}\begin{bmatrix}I & 0\\ (I-B_2\tilde{A}R_{12})^{-1}B_2\tilde{A}R_{11} & (I-B_2\tilde{A}R_{12})^{-1}\end{bmatrix}$$

$$= \begin{bmatrix}R_{11}-R_{12}(I-B_2\tilde{A}R_{12})^{-1}B_2\tilde{A}R_{11} & R_{12}(I-B_2\tilde{A}\,R_{12})^{-1}\\ R_{22}(I-B_2\tilde{A}R_{12})^{-1}B_2\tilde{A}R_{11} & R_{22}(I-B_2\tilde{A}R_{12})^{-1}\end{bmatrix} \quad .$$

Now, let us consider the following feedback:

$$u(t) = -kx(t) = -\left[\tilde{H}^*P_1, 0\right]\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} = -\tilde{H}^*P_1x_1(t) \quad ,$$

then

$$A-Bk = \begin{bmatrix}\tilde{A} & C_2\\ B_2\tilde{A} & B_2C_2+A_2\end{bmatrix} - \begin{bmatrix}\tilde{H}_{6x1}\\ B_2\tilde{H}_{6x1}\end{bmatrix}\left[\tilde{H}^*_{1x6}P_{1_{6x6}} \quad 0\right]$$

$$= \begin{bmatrix}\tilde{A} & C_2\\ B_2\tilde{A} & B_2C_2+A_2\end{bmatrix} - \begin{bmatrix}\tilde{H}\tilde{H}^*P_1 & 0\\ B_2\tilde{H}\tilde{H}^*P_1 & 0\end{bmatrix}$$

$$= \begin{bmatrix}\tilde{A}-\tilde{H}\tilde{H}^*P_1 & C_2\\ B_2(\tilde{A}-\tilde{H}\tilde{H}^*P_1) & B_2C_2+A_2\end{bmatrix} \quad .$$

Define $M = \tilde{A}-\tilde{H}\tilde{H}^*P_1$, then

$$A-Bk = \begin{bmatrix} \tilde{M} & C_2 \\ B_2\tilde{M} & B_2C_2+A_2 \end{bmatrix}, \text{ and } \dot{x}(t) = (A-Bk)x(t) \quad .$$

Note that Bk is only a bounded perturbation of A, hence A-Bk also generates a semigroup. In order to show that

$\int_0^\infty \|x_1(t)\|^p + \int_0^\infty |u(t)|^p dt < \infty$, we will examine the behavior of $R(\lambda;A)\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$ instead.

$$x_1(\lambda) = (R_{11}+R_{12}(I-B_2\tilde{M}R_{12})^{-1}B_2\tilde{M}R_{11})x_1(0)$$

where

$$R_{11} = R(\lambda;\tilde{M}),$$

$$R_{12} = R(\lambda;\tilde{M})C_2R(\lambda;A_2)\left(I-B_2C_2R(\lambda;A_2)\right)^{-1} \quad .$$

Substituting R_{12} into the $x_1(\lambda)$ equation, we get

$$x_1(\lambda) = \Big[R_{11}+R_{11}C_2R(\lambda;A_2)\left(I-B_2C_2R(\lambda;A_2)\right)^{-1}\cdot$$

$$\cdot\left(I-B_2\tilde{M}R_{12}\right)^{-1}B_2\tilde{M}R_{11}\Big]x_1(0)$$

$$= \Big[R_{11}+R_{11}C_2R(\lambda;A_2)((I-B_2\tilde{M}R_{12})(I-B_2C_2R(\lambda;A_2))^{-1}$$

$$B_2\tilde{M}R_{11}\Big]x_1(0)$$

$$=\Big[R_{11}+R_{11}C_2R(\lambda;A_2)\Big\{I-B_2C_2R(\lambda;A_2)-B_2\tilde{M}R_{12}+B_2\tilde{M}R_{12}B_2C_2$$

$$R(\lambda;A_2)\Big\}^{-1}B_2\tilde{M}R_{11}\Big]x_1(0)$$

$$=\Big[R_{11}+R_{11}C_2R(\lambda;A_2)\Big\{I-B_2C_2R(\lambda;A_2)-B_2\tilde{M}R_{12}(I-B_2C_2$$

$$\cdot R(\lambda;A_2)\Big\}^{-1}B_2\tilde{M}R_{11}\Big]x_1(0)$$

$$=\Big[R_{11}+R_{11}C_2R(\lambda;A_2)\Big\{I-B_2C_2R(\lambda;A_2)-B_2\tilde{M}R_{11}C_2R(\lambda;A_2)\Big\}^{-1}$$

$$\cdot B_2\tilde{M}R_{11}\Big]x_1(0)$$

$$=\Big[R_{11}+R_{11}C_2R(\lambda;A_2)\Big(I+B_2+B_2\tilde{M}R_{11})C_2R(\lambda;A_2)$$

$$+((B_2+B_2\tilde{M}R_{11})C_2R(\lambda;A_2))^2+\ldots\Big)B_2\tilde{M}R_{11}\Big]x_1(0)$$

$$=\Big[R_{11}+R_{11}\Big(\hat{B}_2\tilde{M}R_{11}+(\hat{B}_2+\hat{B}_2\tilde{M}R_{11})\hat{B}_2\tilde{M}R_{11}+\ldots\Big)\Big]x_1(0)$$

$$=R_{11}x_1(0)+R_{11}\Big\{I-(\hat{B}_2+\hat{B}_2\tilde{M}R_{11})\Big\}^{-1}\hat{B}_2\tilde{M}R_{11}x_1(0) \qquad (*)$$

Since $\tilde{M}$ is stable, the inverse Laplace transform of $R_{11}x_1(0)$ will be in $L_p^6[0,\infty)$. Moreover the inverse Laplace transform of $I-(\hat{B}_2+\hat{B}_2\tilde{M}R_{11})$ consists of only positively supported delta distributions and L_1-functions, hence it is regular if and only if the determinant of $I-(\hat{B}_2+\hat{B}_2\tilde{M}R_{11})$ is bounded away from 0 on C_+ by the previous lemma. Since $\tilde{M}$ is stable, then $\hat{B}_2+\hat{B}_2\tilde{M}R_{11}$ is bounded away from identity for our aeroelastic system and the determinant condition is clearly satisfied. The second term in (*) is again in $L_p^6[0,\infty)$. This shows that J[u] is indeed finite.

LINEAR OPTIMAL CONTROL THEORY

In recent years, a large amount of literature has been devoted to the L-Q-R problem for infinite dimensional Hilbert space. See, for example [13, 14, 15]. In chapter 4, we have shown that the infinitesimal generator A generates a C_0-semigroup in a reflexive Banach space. The reason for that is because of the aerodynamic energy consideration, i.e., the behavior of $\tilde{Q}(t)$ near origin is of $O(t^{-1/2})$. If we approximate the function $\tilde{Q}(t)$ near origin by $t^{-1/2+\epsilon}$, then the proof for Banach space will go through for Hilbert space $R^6 \times L_2{}^6[0,\infty)$. As a first step, we shall in this section work with the state space $R^6 \times L_2{}^6[0,\infty)$.

First we recall the concepts of stability. Let T(t) denote a strongly continuous semigroup over a Banach space H. If H is finite dimensional then $Re(\sigma(A)) < 0$ is equivalent to $\|T(t)\| \to 0$ or for every x in H, $\|T(t)x\| \to 0$ as $t \to \infty$.

The situation is quite different in the infinite dimensional case. We will give a few different notions of stability.

Definition 5.1. A semigroup $T(t)$ is "exponentially stable" if $\|T(t)\| \to 0$ as $t \to \infty$.

Definition 5.2. A semigroup $T(t)$ is "strongly stable" if $\|T(t)x\| \to 0$ as $t \mapsto \infty$ for every x in H.

Definition 5.3. A semigroup $T(t)$ is "weakly stable" if $[T(t)x,y] \to 0$ as $t \to \infty$ for every x, y in H.

It is easy to show that exponential stability implies strong stability, and strong stability implies weak stability. Note also that $T(t)$ is weakly stable, so is $T^*(t)$. But this is not true for strong stability, just take the left shift semigroup for example. However in finite dimensional cases, these notions are equivalent. Using rational approximation, J.W. Edwards[1] applied the finite dimensional L-Q-R theoty to the flutter problem. However, this problem can not be solved without using the infinite dimensional state space setting as the reason is clear by now.

For our aeroelastic systems, the generator A will have unstable eigenvalues when the speed of aircraft is fast enough.

We will consider first the following L-Q-R problem.

$$\begin{cases} \dot{x} = Ax(t) + Bu(t) \\ z(t) = Cx(t) \end{cases} \tag{5.6}$$

where

$$A = \begin{bmatrix} \tilde{A} & C_2 \\ 0 & A_2 \end{bmatrix} \text{ defined on } R^6 \times L_2{}^6[0,\infty)$$

The cost functional is

$$J[u] = \int_0^\infty [x_1(t),\ x_1(t)] + \int_0^\infty u^2(t)dt$$

where $R = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\lambda > 0$, which, for convenience hereafter will be taken to be unity. Or we can write J[u] as

$$\int_0^\infty [x_1(t),\ x_1(t)] + \int_0^\infty u^2(t)dt \quad .$$

Before stating our main theorms for this section, we shall recall some results of L-Q-R theory in Hilbert spaces (see Balartishnan [12], Gibson[15]).

Theorem 5.2 Let A, B and R be defined as in (5.6), with A the infinitesimal generator of a strongly continuous semigroup. Then there exists a nonnegative, self-adjoint solution of the steady state Ricatti equation if and only if, for each $x \in H$, there is a control for the initial time s and initial state x that $J[u] < \infty$. If $P_\infty(s)$ is such a solution, then $\langle P_\infty(s)x, x\rangle_H = \min J_\infty(s,x,u)$. The optimal control $u(.)$ is given by $u(t) = -B^*P_\infty x(t)$ and $x(t) = T(t)$ $\cdot x(0)$, where $T(t)$ is the strongly continuous semigroup generated by $\hat{A} = A - BB^*P_\infty$.

Proof. We follow the same notation as in Gibson [14]. First we consider a sequence $\{t_n\}$ where $t_n \to \infty$ as $n \to \infty$, and try to investigate the finite time L-Q-R problem for which $t_f = t_n$. For each of these problems, we denote the solution of the Ricatti integral equations by $P_n(.)$ and the cost functional by $J_n(s,x,.)$. An important observation, which follows from the fact that $\min J_n(s,x,u)$ $= \langle P_n(s)x, x\rangle_H$, is that

$$P_n(t) \leq P_m(t) \quad , \quad t_0 \leq t \leq t_n \leq t_m \quad \ldots.(*)$$

Let x and y be in H, we have

$$\langle P_n(t)\, x, y\rangle_H^2 \leq \langle P_n(s)x, x\rangle_H \cdot \langle P_n(s)y, y\rangle_H.$$

Thus

$$J_n(s,x,u_n) = \langle P_n(s)x,\ x\rangle \to J_\infty(s,x,u) \text{ implies that}$$

$$\sup_n |\langle P_n(s)x,\ y\rangle_H| < \infty \text{ for each pair } x \text{ and } y \text{ in } H.$$

Then $\sup_n \|P_n(s)\| < \infty$ by uniform boundedness principle.

Let $\{t_k\}$ be an increasing subsequence of $\{t_n\}$. Then the uniform boundedness of $\|P_k(s)\|$ and (*) imply the existence of a unique self-adjoint operator $P_\infty(s) \in L(H,H)$ such that $P_k(s)x \to P_\infty(s)x$ strongly in H, $x \in H$. Now we have shown that this is true for a subsequence. For the original sequence to hold, just use the generalized Schwarz inequality. Q.E.D.

Theorem 5.3. Suppose there exists a control u(.), such that for each $x \in H$, the cost functional $J[u]$ is finite, and $\lim_{t\to\infty} \|x(t)\| = 0$; i.e., any control drives the state to zero asymptotically. Then there exists at most one uniformly bounded, nonnegative solution of the integral Ricatti equation on $[0,\infty)$.

Proof: Let P(.) be such a solution and define x(t) and u(t) as follows:

$$x(t) = S(t,s)x \quad \text{and} \quad u(t) = -B^*(t)P(t)x(t).$$

Then $\lim_{t\to\infty} \|x(t)\| = 0$ and the uniform boundedness of $\|P(.)\|$ imply $\lim_{t\to\infty} \langle P(t)x(t),\ x(t)\rangle_H = 0$. Then

$$\langle P(s)x, x \rangle_H = J_\infty(s,x,u) \quad .$$

Let ν be an admissible control for s and x, then after some tedious calculations (see Gibson [15]), we can show that

$$J_\infty(s,x,u) = \langle P(s)x, x \rangle_H \leq J_\infty(s,x,\nu) \quad .$$

Thus u is the optimal control for s and x, and $P(s) = P_\infty(s)$, $t_0 \leq s$. .

Q.E.D.

Theorem 5.4. If f is in $L_1[0,\infty)$, g is $L_2[0,\infty)$ then $h = f*g$ exists and h is in $L_2[0,\infty)$.

Proof. See Dunford and Schwarz [16].

Now we can state a theorem for the system (5.6).

Theorem 5.5. If $(\tilde{A},\tilde{H})$ is controllable and (A^*,R) is approximately controllable, then the steady state Ricatti equation $[Ax,Py]+[Px,Ay]+[x,y]-[B^*Px,B^*Py]=0$, $\forall x,y \in D(A)$ has a unique nonnegative self-adjoint solution P_∞, and the optimal feedback control is given by

$u(t) = -B^*P_\infty x(t) \quad .$

Proof. It is a standard result from the finite dimensional control theory that if $(\tilde{A},\tilde{H})$ is controllable, then the finite dimensional Ricatti equation $\tilde{A}^*P_1 + P_1\tilde{A} + I$

$- P_1\tilde{H}\tilde{H}^*P_1 = 0$ has a nonnegative, self-adjoint P_1, such that $\tilde{A} - \tilde{H}\tilde{H}^*P_1$ is stable if $u = -\tilde{H}^*P_1x_1$. This implies that

$$\begin{cases} x_1(t) = e^{(A-\tilde{H}\tilde{H}^*P_1)t}x_1(0) + \int_0^t e^{(\tilde{A}-\tilde{H}\tilde{H}^*P_1)(t-s)}x_2^0(s)ds \\ x_2(t) = T(t)x_2^0 \end{cases}$$

where $T(t)$ is a left-shift semigroup.

Since $(\tilde{A}-\tilde{H}\tilde{H}^*P_1)$ is stable and x_2^0 is an L_2-function, it follows from theorem 5.4. that

$$\int_0^\infty \|x_1(t)\|^2\, dt + \int_0^\infty u^2(t)\, dt < \infty \quad .$$

Moreover, if (A^*,R) is approximately controllable, then $x(t)$ will be weakly stable (see Balakrishnan [13]). But $x_2(t)$ is strongly stable by the fact that $T(t)$ is a left-shift semigroup, even though $x_1(t)$ is only weakly stable. Yet in finite dimensional space, it is the same as strong stability. Hence $\|x(t)\| \to 0$ strongly.

Now by theorems 5.2. and 5.3., the infinite dimensional steady state Ricatti equation has a unique nonnegatives, self-adjoint solution and that there exists an optimal feedback control such that $u(t) = -B^*P_\infty x(t)$ where P_∞ is the nonnegative unique solution of the SSRE.

Q.E.D.

Next, we consider the same L-Q-R problem for

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ z(t) = Cx(t) \end{cases}$$

where

$$A = \begin{bmatrix} \tilde{A} & C_2 \\ B_2\tilde{A} & B_2C_2+A_2 \end{bmatrix}, \quad B = \begin{bmatrix} \tilde{H} \\ B_2\tilde{H} \end{bmatrix} \quad \text{and } C = [I \quad 0] \text{ as before.}$$

The same cost functional

$$J[u] = \int_0^\infty [x_1(t), x_1(t)]\, dt + \int_0^\infty u^2(t)dt \quad .$$

Under the same assumptions as before, we can only prove up to weak stabilizability, but we are only interested in the finite dimensional portions anyway. So the finite dimensional part of the system will be stabilized as before.

CHAPTER 6

SUMMARY, CONCLUSIONS, AND SUGGESTIONS FOR FUTURE RESEARCH

Aerodynamic flutter refers to a subject that has developed from the earliest days of manned flight. It is an unstable motion due to the interaction between structural vibrations and the aerodynamic forces which results in the extraction of energy from the air.

The vital role of flutter control problem is played by the choice of aerodynamic model. The design of active aeroelastic control is greatly aided by the availability of providing a mathematical model valid for arbitrary motions. In the past, engineers used inverse Fourier transform to obtain impulse response function airloads for use in convolution integral solutions of the equations of motion. More common are calculations using finite state, rational function approximations for the unsteady aerodynamic airloads.

In this research, we developed a time-domain model for unsteady aerodynamics. Hence the exact transient response of a three degree-of-freedom airfoil can be obtained.

Then we apply the unsteady aerodynamic models to elastic vehicles. As a first step, we treat the air-foil as a three degree-of-freedom two-dimensional typical sections. The upshot is that the coupled aero-structural dynamics will give rise to an integro-differential equation.

Instead of using rational function approximations, we contend that the problem could not be solved without introducing infinite dimensional state space. We showed then the solution exist, is unique and depends continuously on the initial data via semigroup approach.

The principal advantage of using a semigroup formulation is that once a system has been shown to generate a semigroup, the problem is well-posed immediately. All that is left is the smoothness of the solution. This of course depends on the smooth properties placed on the initial condition and forcing term.

Most of the studies on flutter suppressions are based on finite dimensional L-Q-R theory. The major problem with those approaches is that the control law may not be able to stabilize the original system even though it stabilized the approximating system. Hence we are forced to deal with the complex original system.

First, we consider the input-output stability problem in the reflexive Banach space framework, then we modify the function $\tilde{Q}(t)$ near origin such that $\tilde{Q}(t)$ will be in L_2 $0,\infty)$. Then we use the machinary of L-Q-R theory in Hilbert space to obtain the optimal feedback control law by solving algebraic Ricatti equations in Hilbert space.

Obviously this research needs to be continued in various directions. One extension would be replacing incompressible case to compressible case of the aerodynamics. Another extension can be the replacement of lumped structural dynamic model by distributed parameter system.

Finally, we can see that all the control theories in Hilbert space or Banach space so far do not quite fit our needs for aeroelastic systems. Hence one of the most urgent extensions would be in the control theoretic aspects.

REFERENCES

1. Edwards, J.W., "Unsteady Aerodynamic Modelling and Active Aeroelastic Control," Standard University, SUDAAR 504, Feb. 1977.

2. Edwards, J.W., Ashley, H., and Breakwell, J.V., "Unsteady Aerodynamic Modelling for Arbitrary Motion," AIAA Paper 77-451, March 1977.

3. Bisplinghof, E.L., H. Ashley and R.L. Halfman, Aeroelasticity, Addison-Wesley, 1955.

4. Dowell, E.H. et al, A Modern Course in Aeroelasticity, Sijthoff and Noordhoff, 1978.

5. Sears, W.R., "Operational Methods in the Theory of Airfoils in Non-Uniform Motion", J. of the Franklin Institute, Vol. 230, pp. 95-111, 1940.

6. Burns, J.A., E.M. Cliff and T.L. Herdman, "A State Space Model for an Aeroelastic System," Proceedings of 22nd CDC, 1983. 1074-1077.

7. Balakrishnan, A.V. and J.W. Edwards, "Calculation of the Transient Motion of Elastic Airfoils Forced by Control Surface Motion and Gusts", NASA T.M. 81351, 1980.

8. Sohngen, H., Bestimmung der Auftriebsverteilung fur beliebige instationare Bewegungen (Ebenes Problem), Luftfahrtforschung, Bd. 17, Nr.11 & 12, December, 1940.

9. Theodorson, T., "General Theory of Aerodynamic Instability and the Mechanism of Flutter," NACA Report No. 496, 1935.

10. Balakrishnan, A.V., "Identification and Stochastic Control of Non-Dynamic Systems," Preprints of the 3rd IFAC Symposium on Identification and Parameter Estimation, Hague, June 1973.

11. Pazy, A. "Semigroup of Linear Operators and Applications to Partial Differential Equations" Springer-Verlag, 1983.

12. Balakrishnan, A.V., Applied Functional Analysis, 2nd Ed. Springer-Verlag, 1981.

13. Balakrishnan, A.V. "Strong Stabilizability and the Steady State Ricatti Equation," Applied Mathematics and Optimization, 7 (1981), 335-345

14. Gibson, J.S. "A Note on Stabilization of Infinite Dimensional Linear Oscillations by Compact Linear Feedback," SIAM Control and Optimization Journal, 181 (1980), 311-316.

15. Gibson, J.S., "The Ricatti Integral Equations for Optimal Control Problems on Hilbert Spaces," SIAM Control and Optimization Journal, 17 (1979), 537-565.

16. Dunford, N. and J.T. Schwarz, Linear Operators, II. New York: Wiley, 1963

17. Hille, E. and R.S. Phillips, Functional Analysis and Semigroups, American Mathematical Society, Providence, R.I., 1953.

18. Balakrishnan, A.V., "Active Control of Airfoils in Unsteady Aerodynamics," Applied Mathematics and Optimization, 4 (1978), 171-195.

19. Balakrishnan, A.V., "Optimal Control Problems in Aeroelasticity," Proceedings of the 9th IFIP Conference on System Modelling and Optimization, September 1979, Warsaw, Poland. 1-14.

APPENDIX A

CALCULATION OF THE LIFT

We have in chapter 2 (see Balakrishnan, A.V. and Edwards, J.W. [7]):

$$P=\int_{-1}^{1} P(x,t)dx$$

$$=(-\rho)\left\{\int_{-1}^{1} U\gamma_a(t,x)dx+\frac{\partial}{\partial t}\int_{-1}^{1}\int_{-1}^{x}\gamma_a(t,y)dy\ dx\right\}$$

$$=(-\rho)\left\{U\Gamma(t)-\frac{\partial}{\partial t}\int_{-1}^{1}x\gamma_a(t,x)dx+\Gamma'(t)\right\}.$$

Consider

$$\frac{\partial}{\partial t}\int_{-1}^{1}x\gamma_a(t,x)dx=\int_{-1}^{1}\frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\,x\ dx\int_{-1}^{1}\sqrt{\frac{1+\kappa}{1-\kappa}}$$

$$\cdot\frac{\frac{\partial}{\partial t}w_a(t;\kappa)}{x-\kappa}\,d\kappa-\frac{\partial}{\partial t}\int_{0}^{t}\int_{-1}^{1}\frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\,\frac{x}{2\pi}H(\tau,x)dx$$

$$\cdot\ \Gamma'(t-\tau)d\tau\equiv\underline{T_1+T_2}\ .$$

Now

$$\int_{-1}^{1}\sqrt{\frac{1+\mu}{1-\mu}}\,\frac{\frac{\partial}{\partial t}w_a(t,\mu)}{x-\mu}\,d\mu=\int_{-1}^{1}\sqrt{\frac{1+\mu}{1-\mu}}\,\frac{1}{x-\mu}\,(-h''(t)$$

$$- (\zeta-a)\alpha''(t) - U\alpha'(t)d\zeta + \int_c^1 \sqrt{\frac{1+\zeta}{1-\zeta}}\,\frac{1}{x-\zeta}$$

$$(-(\zeta-c)\beta''(t) - U\beta'(t)d\zeta$$

$$= +\pi h''(t) - (a\pi-\pi-\pi x)\alpha''(t) + U\pi\alpha'(t) + (\cos^{-1}c$$

$$+\sqrt{1-c^2})\beta''(t) + (x-c)(\cos^{-1}c + \sqrt{\frac{1+x}{1-x}}\,\mathrm{Log}\,|\cdot|)\beta''(t)$$

$$+ U\beta'(t)\,(\cos^{-1}c + \sqrt{\frac{1+x}{1-x}}\,\mathrm{Log}\,|\cdot|)$$

where

$$\mathrm{Log}\,|\cdot| = \log\left|\frac{\sqrt{(1-c)(1+x)} + \sqrt{(1+c)(1-x)}}{\sqrt{(1-c)(1+x)} - \sqrt{(1+c)(1-x)}}\right|$$

$$\frac{d}{dx}\,\mathrm{Log}\,|\cdot| = -\frac{\sqrt{1-c^2}}{\sqrt{1-x^2}}\,\frac{1}{x-c}$$

and where we note that

$$\int_{-1}^{1} x(1+x)\sqrt{\frac{1-x}{1+x}}\,dx = 0; \quad \int_{-1}^{1} x\,\log|\cdot|dx = \frac{\pi}{2}\ \ c\ \ 1-c^2;$$

$$\int_{-1}^{1} x(x-c)\,\log\,|\cdot|\,dx = \frac{\pi}{6}\,(1-c^2)^{3/2}; \int_{-1}^{1} x(x-c)$$

$$\cdot\sqrt{\frac{1-x}{1+x}}\,dx = \frac{\pi}{2}(1+c); \quad \int_{-1}^{1} x\sqrt{\frac{1-x}{1+x}}\,dx = -\frac{\pi}{2} \quad .$$

Hence

$$T_1 = -\pi h''(t) + (a\pi)\alpha''(t) - U\pi\alpha'(t) + (c\cos^{-1}c - \sqrt{1-c^2}$$

$$+ \frac{1}{3}(\sqrt{1-c^2})^3)\beta''(t) + U(c\sqrt{1-c^2} - \cos^{-1}c)\beta'(t)\ .$$

Next

$$T_2 = \frac{\partial}{\partial t}\int_0^t \int_{-1}^1 \frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\, x\, (-\frac{1}{2\pi})\, H(\gamma,x)dx\, \Gamma'(t-\gamma)d\gamma$$

$$= \frac{\partial}{\partial t}\int_0^t (1 + U\zeta - \sqrt{(1+U\zeta)^2-1})\Gamma'(t-\zeta)d\zeta\ .$$

Hence

$$P = (-\rho)\ U\Gamma(t) + U\pi\alpha'(t) - U(c\sqrt{1-c^2} - \cos^{-1}c)\beta'(t)$$

$$+ \pi h''(t) - a\pi\alpha''(t) - (\frac{1}{3}(\sqrt{1-c^2})^3 + c\cos^{-1}c$$

$$- \sqrt{1-c^2})\beta''(t) - \frac{d}{dt}\int_0^t c_3(t-\delta)[B,\dot{Z}(\delta)]d\delta$$

where

$$C_3(t) = \int_0^t c_1(t-\delta)\left(U\delta - \sqrt{U^2\delta^2 + 2U\delta}\right)d\delta\ .$$

APPENDIX B

CALCULATION OF THE PITCHING MOMENT M_α

We have:

$$M_\alpha = (-\rho)\left(\int_{-1}^{1}(x-a)U\gamma_a(t,x)dx + \frac{\partial}{\partial t}\int_{-1}^{1}(x-a)\int_{-1}^{x}\gamma_a(t,y)dy\right)$$

$$= (-\rho)\left(\int_{-1}^{1}U(x-a)\gamma_a(t,x)dx - \frac{\partial}{\partial t}\int_{-1}^{1}\frac{(x-a)^2}{2}\gamma_a(t,x)dx\right.$$

$$\left. + \Gamma'(t)\frac{(1-a)^2}{2}\right),$$

$$\int_{-1}^{1}(x-a)\gamma_a(t,x)dx = -a\Gamma(t) + \int_{-1}^{1}x\gamma_a(t,x)dx ,$$

$$\int_{-1}^{1}x\gamma_a(t,x)dx = \int_{-1}^{1}\frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\,x\left(\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\frac{w_a(t,\xi)}{x-\xi}d\xi\right.$$

$$\left. - \frac{1}{2\pi}\int_0^t H(\theta,x)\Gamma'(t-\theta)d\theta\right)dx$$

$$= \int_{-1}^{1}\frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\,x\,dx\int_{-1}^{1}\sqrt{\frac{1+\eta}{1-\eta}}\frac{w_a(t,\eta)}{x-\eta}d\eta + \int_0^t((1+U\eta)$$

$$-\sqrt{U^2\eta^2+2U\eta})\Gamma'(t-\eta)d\eta$$

$$= T_1 + (\Gamma(t) - \Gamma(0)) + \int_0^t c_3(t-\upsilon)[B,\dot{Z}(\upsilon)]d\upsilon .$$

$$T_1 = \int_{-1}^{1} \frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\, x \Big\{ \int_{-1}^{1} \sqrt{\frac{1+\eta}{1-\eta}} \{-h'(t) - (\eta - a)\alpha'(t)$$

$$-U\alpha(t)\} \frac{d\eta}{x-\eta}$$

$$+ \int_{c}^{1} \sqrt{\frac{1+\eta}{1-\eta}} \Big\{ -(\eta - c)\beta'(t) - U\beta(t) \Big\} \frac{d\eta}{x-\eta} \Big\} dx$$

$$= \int_{-1}^{1} \frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\, x\ \{ \pi(h'(t) + U\alpha(t)) - \pi(a-1-x)\alpha'(t)$$

$$+ U\ (\cos^{-1}c + \sqrt{\frac{1+x}{1-x}} \log|\cdot|)\beta(t) + \Big\{ (\cos^{-1}c + \sqrt{1-c^2})$$

$$+ (\cos^{-1}c + \frac{1+x}{1-x} \log|\cdot|)(x-c) \Big\} \beta'(t) \}\, dx$$

$$= -\pi(h'(t) + U\alpha(t)) + \pi a \alpha'(t) + U(-\cos^{-1}c + c\sqrt{1-c^2})$$

$$\cdot \beta(t) + \Big\{ c \cos^{-1}c - \frac{1}{3}(2+c^2)\sqrt{1-c^2} \Big\} \beta'(t)\ .$$

Next, the "non-circulatory" terms (that is, terms not containing $\Gamma(t)$) in:

$$\frac{\partial}{\partial t} \int_{-1}^{1} \frac{(x-a)^2}{2}\ \gamma_a(t,x)dx$$

$$= \int_{-1}^{1} \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}\,(x-a)^2 \Big\{ \pi(h''(t) + U\alpha'(t)) - \pi(a-1-x)$$

$$\cdot \alpha''(t) + (\cos^{-1}c + \sqrt{\frac{1+x}{1-x}} \log|\cdot|)\beta'(t) + (\cos^{-1}c$$

$$+\sqrt{\frac{1+x}{1-x}}\log\,|\cdot|\ )(x-c)\beta''(t)$$

$$= (\tfrac{1}{2}+a^2+a)\pi(h''(t)+U\alpha'(t)) + \pi(\tfrac{1}{8}-\tfrac{a}{2}-a^3-\tfrac{a^2}{2})\alpha''(t) + U\Big\{(\tfrac{1}{2}$$

$$+a+a^2)\cos^{-1}c + (\tfrac{1}{6}+2a^2+\tfrac{2c^2}{3}-2ac)\sqrt{1-c^2}\Big\}\beta'(t) + \Big\{(\cos^{-1}c$$

$$+\sqrt{1-c^2}(\tfrac{1}{2}+a^2+a) + \tfrac{1}{12}\sqrt{1-c^2}(4ac^2-6a^2c-c^3-4a-\tfrac{c}{2})$$

$$-\cos^{-1}c(a+\tfrac{a^2}{2}+\tfrac{3}{8}+\tfrac{c}{2}+a^2c)\Big\}\beta''(t)\ .$$

Hence

$$M_\alpha = (-\rho)\Big\{-aU\Gamma(t) + U\{\Gamma(t) - \Gamma(0)\} + U\int_0^t c_3(t-\eta)$$

$$[B,\dot{Z}(\eta)]d\eta + U\{-\pi h'(t) - U\alpha(t)\pi + \pi a\alpha'(t)$$

$$+ U\,(-\cos^{-1}c+c\sqrt{1-c^2})\beta(t) + (c\cos^{-1}-\tfrac{1}{3}(2+c^2)\sqrt{1-c^2}$$

$$\beta'(t)\} + \frac{(1-a)^2}{2}\ \Gamma'(t) - (\tfrac{1}{2}+a^2+a)\pi)h''(t)+U\alpha'(t))$$

$$-\pi(\tfrac{1}{8}-\tfrac{a}{2}-a^3-\tfrac{a^2}{2})\alpha''(t) - U\{(\tfrac{1}{2}+a+a^2)\cos^{-1}c + (\tfrac{1}{6}+2a^2$$

$$+\tfrac{2c^2}{3}-2ac)\sqrt{1-c^2}\}\beta'(t) + \{(\cos^{-1}c\ (\tfrac{1}{8}+\tfrac{a^2}{2}-\tfrac{c}{2}-a^2c-ac)$$

$$+\sqrt{1-c^2}\ (\tfrac{1}{2}+a^2+\tfrac{2}{3}a+\tfrac{1}{3}ac^2-\tfrac{1}{2}a^2c-\tfrac{c^3}{12}-\tfrac{c}{24})\}\beta''(t) +$$

$$+\frac{d}{dt}\frac{1}{2}\int_0^t\left\{\frac{1}{\pi^2}\int_{-1}^{1}(x-a)^2\sqrt{\frac{1-x}{1+x}}\,H(\upsilon,x)dx\right\}\Gamma'(t-\upsilon)d\upsilon\Big\}$$

where the factor in square brackets in the integrand in the last term is

$$\frac{1}{\pi^2}\int_0^1(x-a)^2\sqrt{\frac{1-x}{1+x}}\,H(\upsilon,x)dx$$

$$=(\frac{1}{2}\sqrt{\frac{z+1}{z-1}}-z^2+z\sqrt{z^2-1})+2a(z-\sqrt{z^2-1})$$

$$+a^2(\sqrt{\frac{z+1}{z-1}}-1)\,.$$

It is also possible to split the circulatory and non-circulatory terms in yet another way. Thus by not splitting

$$\int_{-1}^{1}(x-a)\gamma_a(t,x)dx=-a\Gamma(t)+\int_{-1}^{1}x\gamma_a(t,x)dx$$

but directly calculating the left side, we have:

$$\int_{-1}^{1}(x-a)\gamma_a(t,x)dx=T_3+T_4$$

$$T_3=\int_{-1}^{1}\frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}\,(x-a)\{\pi h'(t)+\pi U\alpha(t)+\pi(1-x-a)\alpha'(t)+$$

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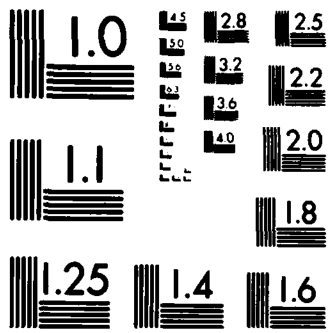

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$$+ U(\cos^{-1}c + \sqrt{\frac{1+x}{1-x}}\log|\cdot|)\beta(t) + \{(\cos^{-1}c + \sqrt{1-c^2})$$

$$+ (\cos^{-1}c + \sqrt{\frac{1+x}{1-x}}\log|\cdot|)(x-c)\}\beta'(t)\Big\}dx$$

$$= -(1+2a)\pi U\alpha(t) - (1+2a)\pi h'(t) + 2a^2\pi\alpha'(t) + u((-1$$

$$-2a)\cos^{-1}c + (c-2a)\sqrt{1-c^2})\beta(t) + \{(\cos^{-1}c + \sqrt{1-c^2})$$

$$\cdot(-1-2a) + 2\cos^{-1}c\,(ac+\tfrac{a}{2}+\tfrac{c}{2}+\tfrac{1}{2}) + \tfrac{1}{3}\sqrt{1-c^2}(1-c^2+3ac)\}$$

$$\cdot\,\beta'(t)$$

$$T_4 = +\,2\pi\int_0^t\{z-\sqrt{z^2-1} + a\sqrt{\frac{z-1}{z-1}} - a\}\Gamma'(t-\theta)d\theta.$$

This yields:

$$M_\alpha = (-\rho)\Big\{U^2(-1-2a)\pi\alpha(t) + U^2\{(-1-2a)\cos^{-1}c$$

$$+ (c-2a)\sqrt{1-c^2}\}\beta(t) - U(1+2a)\pi h'(t) + 2Ua^2\pi\alpha'(t)$$

$$+ U\{\tfrac{1}{3}\sqrt{1-c^2}(1-c^2+3ac) + 2\cos^{-1}c(ac+\tfrac{a}{2}+\tfrac{c}{2}+\tfrac{1}{2}) -(1+2a)$$

$$(\cos^{-1}c + \sqrt{1-c^2})\}\beta'(t) - (\tfrac{1}{2}+a^2+a)\pi U\alpha'(t)$$

$$- U\{(\tfrac{1}{2}+a+a^2)\cos^{-1}c + (\tfrac{1}{6}+2a^2+\tfrac{2c^3}{3}-2ac)\sqrt{1-c^2}\}\beta'(t)-$$

$$- (\tfrac{1}{2}+a^2+a)\pi h''(t) - \pi(\tfrac{1}{8}-\tfrac{a}{2}-a^3-\tfrac{a^2}{2})\alpha''(t) - \{(\cos^{-1}c$$

$$+ \sqrt{1-c^2})(\tfrac{1}{2}+a^2+a) + \tfrac{1}{12}\sqrt{1-c^2}(4ac^2-6a^2c-c^3-4a-\tfrac{c}{2})$$

$$- (\cos^{-1}c)(a+\tfrac{a^2}{2}+\tfrac{3}{8}+\tfrac{c}{2}+a^2c+ac)\}\beta''(t) + \frac{(1-a)^2}{2}\Gamma'(t)$$

$$+ U\int_0^t (z-\sqrt{z^2-1}+a\sqrt{\frac{z+1}{z-1}}-a)\Gamma'(t-\theta)d\theta + \frac{1}{2}\frac{d}{dt}\int_0^t\{\frac{1}{2}$$

$$\sqrt{\frac{z+1}{z-1}}-z^2+z\sqrt{z^2-1}+2a(z-\sqrt{z^2-1})+ a^2(\sqrt{\frac{z-1}{z-1}}-1)\ \Gamma'(t-\theta)\}d\theta$$

$$M_\alpha = (-\rho)\Big\{U^2(-1-2a)\pi\alpha(t) + U^2\Big\{(-1-2a)\cos^{-1}c + (c-2a)$$

$$\sqrt{1-c^2}\Big\}\beta(t) - U(1+2a)\pi h'(t) + U\pi(a^2-a-\tfrac{1}{2})\alpha'(t)$$

$$+ U\Big\{(-\tfrac{5}{6}+3ac-2a-2a^2-\tfrac{c^2}{3}-\tfrac{2c^3}{3})\sqrt{1-c^2} + (2ac+c-\tfrac{1}{2}-2a-a^2)$$

$$\cos^{-1}c\Big\}\beta'(t) - \Big\{(\tfrac{1}{8}-\tfrac{a^2}{2}-\tfrac{c}{2}-a^2c-ac)\cos^{-1}c + \sqrt{1-c^2}(\tfrac{1}{2}+a^2$$

$$+\tfrac{2}{3}a+\tfrac{1}{3}ac^2-\tfrac{a^2c}{2}-\tfrac{c^3}{12}-\tfrac{c}{24})\Big\}\beta''(t) + \frac{(1-a)^2}{2}\Gamma'(t)$$

$$+ U\int_0^t (z-\sqrt{z^2-1} + a\sqrt{\frac{z+1}{z-1}}-a)\Gamma'(t-\rho)d\rho +$$

$$+\frac{d}{dt}\int_0^t \frac{1}{2}\Big\{\frac{1}{2}\sqrt{\frac{z+1}{z-1}} - z^2 + z\sqrt{z^2-1} + 2a(z-\sqrt{z^2-1})$$

$$+ a^2\Big(\frac{\sqrt{z+1}}{\sqrt{z-1}}-1\Big)\Big\}\Gamma'(t-\rho)d\rho\Big\} .$$

The last two lines may be written as

$$\frac{\Gamma'(t)}{2} + U\int_0^t c_4(t-\rho)[B,Z(\rho)]\,d\rho + Ua\int_0^t(1-c_2(t-\rho))$$

$$[B,(\rho)]\,d\rho + \frac{d}{dt}\Big\{\frac{1}{4}\int_0^t(1-c_2(t-\rho)[B,Z(\rho)]d\rho + \frac{1}{4}\Gamma(t)$$

$$+\frac{1}{2}\int_0^t c_5(t-\rho)[B,Z(\rho)]\,d\rho + a\int_0^t c_3(t-\rho)[B,Z(\rho)]\,d\rho$$

$$+\frac{a^2}{2}[B,Z(t)]\Big\}$$

$$=\int_0^t\Big\{\frac{3}{4}c_1(t-\rho) + Uc_4(t-\rho) + Ua(1-c_2(t-\rho))\Big\}[B,Z(\rho)]\,d\rho$$

$$+\frac{d}{dt}\int_0^t\Big\{\Big(\frac{1}{4}+\frac{a^2}{2}\Big) - \frac{1}{4}c_2(t-\rho) + ac_3(t-\rho) + \frac{1}{2}c_5(t-\rho)\Big\}$$

$$\cdot[B,Z(\rho)]\,d\rho$$

where

$$c_4(t)=\int_0^t c_1(t-\rho)(1-U\rho-\sqrt{U^2\rho^2+2U\rho})\,d\rho$$

$$= c_2(t) + c_3(t)$$

$$c_5(t) = \int_0^t c_1(t-\rho)\left((1-U\rho)\sqrt{U^2\rho^2-2U\rho}-(1-U\rho)^2\right)d\rho$$

APPENDIX C

CALCULATION OF THE FLAP MOMENT M_β

We have:

$$M_\beta = \int_c^1 (x-c)P(x,t)dx$$

$$= (-\rho)\left\{\int_c^1 U(x-c)\gamma_a(t,x)dx + \frac{\partial}{\partial t}\int_c^1 (x-c)dx \int_{-1}^1 \gamma_a(t,y)dy\right\}$$

$$= (-\rho)\left\{\int_c^1 U(x-c)\gamma_a(t,x)dx + \frac{\partial}{\partial t}\,\frac{(1-c)^2}{2}\int_{-1}^1 \gamma_a(t,y)dy\right.$$

$$\left. - \frac{\partial}{\partial t}\int_c^1 \frac{(x-c)^2}{2}\gamma_a(t,x)dx\right\}$$

$$= (-\rho)\left\{U\int_c^1 (x-c)\gamma_a(t,x)dx + \frac{(1-c)^2}{2}\Gamma'(t) - \frac{1}{2}\cdot\right.$$

$$\left.\cdot\frac{\partial}{\partial t}\int_c^1 (x-c)^2\gamma_a(t,x)dx\right\}.$$

Now

$$\int_{-1}^1 \sqrt{\frac{1+\upsilon}{1-\upsilon}}\,\frac{w_a(t,\upsilon)}{x-\upsilon}\,d\upsilon$$

$$= \int_{-1}^1 \sqrt{\frac{1+\upsilon}{1-\upsilon}}\,\frac{(-h'(t)-(\upsilon-a)\alpha'(t)-U\alpha(t))}{(x-\upsilon)}\,d\upsilon \; +$$

$$+\int_c^1 \sqrt{\frac{1+\mu}{1-\mu}}\ \frac{\dot{\zeta}-(\mu-c)\beta'(t)-U\beta(t)d\mu}{(x-\mu)}.$$

The first term is equal to

$$\pi h'(t) - \pi(a-1-x)\alpha'(t) + U\pi\alpha(t).$$

Next let

$$\frac{2}{\pi}\int_c^1 (x-c)\sqrt{\frac{1-x}{1+x}}\,dx \int_0^1 \sqrt{\frac{1+\delta}{1-\delta}}\,d\delta = F_1(c)$$

$$\frac{2}{\pi}\int_c^1 (x-c)\sqrt{\frac{1-x}{1+x}}\int_c^1 \sqrt{\frac{1+\delta}{1-\delta}}\,\frac{\delta}{x-\delta}\,d\delta = F_2(c).$$

Now let us define

$$H_1(\sigma) = \frac{2}{\pi}\int_c^1 (x-c)\sqrt{\frac{1-x}{1+x}}\,H(\sigma,x)dx$$

$$= (-2)\int_c^1 \frac{(x-c)}{x-z}\sqrt{\frac{1-x}{1+x}}\,dx\sqrt{\frac{z+1}{z-1}}$$

$$= (-2)\sqrt{\frac{z+1}{z-1}}\left\{\int_c^1 \sqrt{\frac{1-x}{1+x}}\left(1-\frac{z-c}{x-z}\right)dx\right\}$$

$$= (-2)\sqrt{\frac{z+1}{z-1}}\left\{\cos^{-1}c - \sqrt{1-c^2} - (z-c)\cos^{-1}c\right.$$

$$\left. + \frac{2(z-1)(z-c)}{\sqrt{z^2-1}}\tan^{-1}\frac{\sqrt{(1-c)}\sqrt{z+1}}{\sqrt{(1+c)}\sqrt{z-1}}\right\}.$$

Hence

$$\int_c^1 (x-c)\gamma_a(t,x)dx$$

$$= \frac{2}{\pi}\int_c^1 (x-c)\sqrt{\frac{1-x}{1+x}}(\pi h'(t)+\pi(x-1)\alpha'(t)+a\pi\alpha'(t)$$

$$+U\pi\alpha(t))dx-(F_2(c)-cF_1(c))\beta'(t)-UF_1(c)\beta(t)$$

$$-\frac{1}{2\pi}\int_0^t H_1(\eta)\Gamma'(t-\eta)d\eta$$

$$= 2(h'(t)-a\alpha'(t)+U\alpha(t))((1+\frac{c}{2})\sqrt{1-c^2}-(c+\frac{1}{2})\cos^{-1}c)$$

$$+\left\{(\frac{2}{3}-\frac{c^2}{3})\sqrt{1-c^2}-\cos^{-1}c\right\}\alpha'(t)-(F_2(c)-cF_1(c))\beta'(t)$$

$$- UF_I(c)\beta(t) - \frac{1}{2\pi}\int_0^t H_1(\gamma)\Gamma'(t-\gamma)d\gamma.$$

Next

$$\int_c^1 (x-c)^2\gamma_a(t,x)dx$$

$$= \frac{2}{\pi}\int_0^1 (x-c)^2\sqrt{\frac{1-x}{1+x}}(\pi h'(t)-a\pi\alpha'(t)+U\pi\alpha(t))dx$$

$$+ 2\int_c^1 (x-c)^2\sqrt{\frac{1-x}{1+x}}(x+1)dx\,\alpha'(t) - (g_2(c)-cg_1(c))\cdot$$

$$\cdot \beta'(t) - Ug_1(c)\beta(t) - \frac{1}{2\pi}\int_0^t H_2(\zeta)\Gamma'(t-\zeta)d\zeta$$

where

$$H_2(\sigma) = \frac{2}{\pi}\int_c^1 (x-c)^2 \sqrt{\frac{1-x}{1+x}}\, H(\sigma,x)dx$$

$$g_1(c) = \frac{2}{\pi}\int_c^1 (x-c)^2 \sqrt{\frac{1-x}{1+x}}\, dx \int_c^1 \sqrt{\frac{1+\xi}{1-\xi}}\, \frac{1}{x-\xi}\, d\xi$$

$$g_2(c) = \frac{2}{\pi}\int_c^1 (x-c)^2 \sqrt{\frac{1-x}{1+x}}\, dx \int_c^1 \sqrt{\frac{1+\eta}{1-\eta}}\, \frac{\eta}{x-\eta}\, d\eta .$$

First term is equal to

$$2h_1(c)(h'(t) - a\alpha'(t) + U\alpha(t)).$$

Second term is equal to

$$2h_2(c)\alpha'(t).$$

Next

$$H_2(\sigma) = \frac{2}{\pi}\int_c^1 (x-c)^2 \sqrt{\frac{1-x}{1+x}}\, H(\sigma,x)dx$$

$$= (-2)\left(\int_c^1 \frac{(x-c)^2}{x-z} \sqrt{\frac{1-x}{1+x}}\, dx\right) \sqrt{\frac{z+1}{z-1}}$$

$$= (-2)\sqrt{\frac{z-1}{z-1}} \int_c^1 \sqrt{\frac{1-x}{1+x}}\;\; \frac{(z-c)^2}{x-z}\left\{-x-2x+z\right\}dx$$

$$= (-2)\sqrt{\frac{z+1}{z-1}}\Big\{(z-c)^2(-\cos^{-1}c+2\frac{z-1}{\sqrt{z^2-1}}\tan^{-1}\sqrt{\frac{(1-c)}{(1+c)}\frac{z+1}{z-1}})$$

$$+(z-2c)(\cos^{-1}c-\sqrt{1-c^2})+(\sqrt{1-c^2})-\tfrac{1}{2}\cos^{-1}c-\frac{c\sqrt{1-c^2}}{2}\Big\}$$

$$= (-4)(z-c)^2\tan^{-1}\frac{\sqrt{(1-c)}}{\sqrt{(1+c)}}\frac{\sqrt{z+1}}{\sqrt{z-1}}-2\sqrt{\frac{z+1}{z-1}}\Big\{(z-2c-(z-c)^2-\tfrac{1}{2})$$

$$\cos^{-1}c + (1+2c-\tfrac{c}{2}-z)\sqrt{1-c^2}\Big\} .$$

Hence putting it altogether we have:

$$\frac{M_\beta}{(-\rho)} = U((2+c)\sqrt{1-c^2}-(2c+1)\cos^{-1}c)(h'(t)-a\alpha'(t)+$$

$$+ U\alpha(t))$$

$$+U\Big\{(\tfrac{2}{3}+\tfrac{c^2}{3})\sqrt{1-c^2} - c\cos^{-1}c\Big\}\alpha'(t)$$

$$-U(F_2(c)-cF_1(c))\beta'(t)-U^2F_1(c)\beta(t)$$

$$-\frac{U}{2\pi}\int_0^t H_1(\zeta)\Gamma'(t-\zeta)d\zeta+\frac{(1-c)^2}{2}\Gamma'(t)-h_1(c)(h''(t)$$

$$-a\alpha''(t)+U\alpha'(t))-h_2(c)\alpha''(t)+(\frac{g_2(c)-cg_1(c)}{2})\beta''(t)$$

$$+\frac{U}{2}g_1(c)\beta'(t)+\frac{1}{2}(\frac{1}{2\pi})\frac{\partial}{\partial t}\int_0^t H_2(\upsilon)\Gamma'(t-\upsilon)d\upsilon .$$

$$= (-\rho)\, U^2 \Big\{(2+c)\sqrt{1-c^2}-(2c+1)\cos^{-1}c\Big\}\alpha(t)-U^2 f_1(c)\beta(t)$$

$$+U\Big\{(2+c)\sqrt{1-c^2}-(2c+1)\cos^{-1}c\Big\}h'(t)+U\Big\{(\tfrac{8}{3}+c+\tfrac{c^2}{3})\sqrt{1-c^2}$$

$$-(1+3c)\cos^{-1}c-h_1(c)\Big\}\alpha'(t)+U\Big\{\frac{g_1(c)}{2}-(f_2(c)-cf(c)\Big\}\beta'(t)$$

$$-h_1(c)h''(t)+(ah_1(c)-h_2(c))\alpha''(t)+\tfrac{1}{2}(g_2(c)-cg_1(c))\beta''(t)$$

$$-\frac{U}{2\pi}\int_0^t H_1(\upsilon)\Gamma'(t-\upsilon)d\upsilon+\frac{1}{2}\,\frac{1}{2\pi}\,\frac{d}{dt}\int_0^t H_2(\upsilon)\Gamma'(t-\upsilon)d\upsilon\Big\} \, .$$

$$h_1(c)=\int_0^1\sqrt{\frac{1-x}{1+x}}\,(x-c)^2dx$$

$$=(\frac{1+2c+2c^2}{2})\cos^{-1}c - \frac{1}{6}(2c^2+9c+4)\sqrt{1-c^2} \, .$$

$$h_2(c)= \int_0^1\sqrt{\frac{1-x}{1+x}}\,(x+1)(x-c)^2dx$$

$$=(\frac{c^2}{2}+\frac{1}{8})\cos^{-1}c - (\frac{c^3}{12}+\frac{13}{24}c)\sqrt{(1-c^2)} \, .$$

APPENDIX D
EQUATIONS OF MOTION

The equations of motion of typical section shown in fig. 2.1 are derived from Lagrange's equations (see Edwards, J.W. [1])

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}(T-V) + \frac{\partial}{\partial q_i}(T-V) = Q_i$$

where the Kinetic energy is

$$T = \frac{1}{2}\int_{-1}^{1} \dot{z}^2 \rho(x)\, dx \quad .$$

The airfoil deflection is

$$z_a = -h-(x-a)\alpha - (x-c)\beta U(x-c) \quad .$$

The potential energy V is

$$V = \frac{1}{2}(k_h h^2 + k_\alpha \alpha^2 + k_\beta \beta^2) \quad .$$

Thus

$$T = \frac{1}{2}\left\{m \quad \dot{h}^2 + I_\alpha \dot{\alpha}^2 + I_\beta \dot{\beta}^2\right\} + S_\alpha \dot{h}\dot{\alpha} + S_\beta \dot{h}\dot{\beta} + \left\{I_\beta + \ (c-a)\right\}\dot{\alpha}\,\dot{\beta} \, .$$

Substituting T and V into the Lagrange's equation, we get the equations of motion:

$$m\ddot{h} + S_\alpha \ddot{\alpha} + S_\beta \ddot{\beta} + k_h h = P$$

$$S_\alpha \ddot{h} + I_\alpha \ddot{\alpha} + \left\{ I_\beta + S_\beta (c-a) \right\} \ddot{\beta} + k_\alpha \alpha = M_\alpha$$

$$S_\beta \ddot{h} + \left\{ I_\beta + S_\beta (c-a) \right\} \ddot{\alpha} + I_\beta \ddot{\beta} + k_\beta \beta = M_\beta$$

Let

$$x = \begin{bmatrix} h \\ \alpha \\ \beta \end{bmatrix} , \quad L = \begin{bmatrix} P \\ M_\alpha \\ M_\beta \end{bmatrix} .$$

Then the equations above can be written as:

$$M_s \ddot{x} + B_s \dot{x} + K_s x = \frac{L}{m} ,$$

where

$$M_s = \begin{bmatrix} 1 & x_\alpha & x_\beta \\ x_\alpha & r_\alpha^2 & r_\beta^2 + x_\beta (c-a) \\ x_\beta & r_\beta^2 + x_\beta (c-a) & r_\beta^2 \end{bmatrix}$$

$$K_s = \begin{bmatrix} \omega_h^2 & 0 & 0 \\ 0 & r_\alpha^2 \omega_\alpha^2 & 0 \\ 0 & 0 & r_\beta^2 \omega_\beta^2 \end{bmatrix}$$

$$B_s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 r_\beta^2 \zeta_\beta \omega_\beta \end{bmatrix} .$$

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